On the Design of Maximal-Rate Shape-Preserving 2×2 and 3×3 Space-Time Codes for Noncoherent Energy-Detection Based PPM Communications

Chadi Abou-Rjeily, *Senior Member IEEE*

***Abstract*—In this paper, we propose novel 2×2 and 3×3 noncoherent Space-Time (ST) codes for Impulse-Radio Ultra-Wideband (IR-UWB) communications with Pulse Position Modulation (PPM). The code design is based on existing coherent ST codes where the associated constellation is confined in order to achieve full diversity with analog energy detectors. A comprehensive performance analysis guides this confinement and highlights the maximum rates that can be achieved by this specific layered codeword structure with two and three transmit antennas. The proposed codes attain these maximum rates without introducing any expansion to the unipolar PPM signal set.**

***Index Terms*—Pulse Position Modulation, PPM, Space-Time, ST, Noncoherent, Energy Detection, UWB, MIMO.**

I. Introduction

Noncoherent communication techniques are appealing in scenarios where it is hard to estimate the underlying communication channel. A typical practical application corresponds to Impulse-Radio Ultra-Wideband (IR-UWB) systems where the energy of the transmitted sub-nanosecond pulses is spread over hundreds of multi-path components. In this context, Transmitted Reference (TR) systems [1], [2] and Energy Detection (ED) systems [3]–[7] constitute viable low-cost and power-efficient alternatives to the coherent Rake and matched-filter receivers. Moreover, given the excessively large bandwidth of UWB signals that extends over several GHz, the sampling frequencies are prohibitively high and, consequently, the efficiency and power-consumption of the analog-to-digital converters (ADC) constitute the big hurdle in the implementation of coherent IR-UWB receivers [8]. In this context, the noncoherent TR/ED schemes bypass the heavy ADC where the delay-lines/integrators are implemented in the analog domain.

We consider IR-UWB communications with Pulse Position Modulation (PPM) and ED-based receivers. PPM is advantageous because it is easier to control the delays of the UWB pulses rather than controlling their amplitudes and phases [9]. On the other hand, energy detection constitutes an attractive solution that results in limited hardware complexity. In ED systems, the received signal is first filtered then squared and integrated where such processing takes place in the analog domain. At a second time, the ED receiver decides in favor of the PPM slot containing the maximum energy. While the problem of noncoherent IR-UWB communications with PPM and analog energy detectors is well explored in the literature, this problem was considered almost exclusively in the context of Single-Input-Single-Output (SISO) systems [3]–[7].

The author is with the Department of Electrical and Computer Engineering of the Lebanese American University (LAU), Byblos, Lebanon. (e-mail: chadi.abourjeily@lau.edu.lb).

In addition to the conventional SISO IR-UWB communications, the spatial dimension was explored in numerous contributions that targeted the Space-Time (ST) code design for Multiple-Input-Multiple-Output (MIMO) IR-UWB systems [10]–[20]. The existing literature covers coherent ST codes for IR-UWB with PPM [10]- [11] and Pulse Amplitude Modulation (PAM) [12]- [13]. Noncoherent ST code designs for differential TR IR-UWB communications were proposed and analyzed in [14] and in [15]–[19] for PPM and PAM, respectively. More recently, the first known noncoherent ST scheme for PPM with analog energy detection was suggested in [20].

Despite the huge literature on ST coding with IR-UWB, none of the existing codes in [10]–[19] can be taken off the shelf and applied with ED-based PPM communications for the following reason. While the detection of these codes is based on the knowledge of the samples of the received signal whether for coherent communications [10]–[13] or differential communications [14]–[19], these samples are not available at the analog ED-based receiver that bypasses the ADC. In this context, the energies collected in the PPM slots constitute the only available decision metrics on which the detection process is based. In this context, only the noncoherent PPM code in [20] is amenable to ED. This focal constraint of decodability with analog ED-based receivers drastically differentiates the ST code design approach in [10]–[19] as compared to [20]. In fact, while the coherent codes [10]–[13] and differential codes [14]–[19] are all designed based on the design criteria in [21], it has been proven in [20] that these design criteria do not hold for ED-PPM-ST codes. In this context, two better suited design criteria for ED were derived in [20]. These criteria hold for the permutation-based layered codeword structure that was adopted for the design of coherent and noncoherent PPM-ST codes in [11] and [20], respectively.

In this paper, we consider the problem of noncoherent ED-PPM-ST code design in the cases of two and three transmit antennas. The proposed codes are closely related to the codes in [20] where the newly proposed schemes and [20] share the following key construction constraints. (i): Compliance to analog ED with no ADC. (ii): PPM shaping constraint where the ST codes do not introduce any expansion to the uncoded real-valued and unipolar PPM signal set. These two constraints render the proposed schemes an appropriate solution to low-cost carrier-less UWB communications. At the receiver side, the simple ED-based detection is applied where the ST coding is entirely based on the integrated signal energy. While all of the above aspects are common to the proposed schemes and [20], the main differentiators are as follows. While the codes in [20] can be applied with an arbitrary number of transmit antennas, the proposed ST codes are limited to the cases of two and three antennas. Nevertheless, the superiority of the proposed codes resides in the achievable data rates that exceed those attained by [20] with M-ary PPM for all values of M. In this context, we keep the same codeword structure as in [20] and we solve for the codebook with maximum cardinality to be associated with these codewords.

This work starts by reformulating the design criteria in [20] thus fixing the structure of the noncoherent ED-PPM-ST codebook based on excluding some elements from the codebook if other elements are to be included. Following from this formulation, we prove that the maximum attainable codebook cardinalities are $\frac{M(M-2)}{2}$ and $M\left[M-4+\left\lfloor\frac{(M-5)(M-6)}{6}\right\rfloor\right]$ in the cases of two and three transmit antennas, respectively. The provided analysis guides the search for the codebooks with maximum cardinalities where we provide the explicit expressions of several equivalent codebooks that achieve these maximum cardinalities. This advantageous rate translates into enhanced coding gains in the average-to-large ranges of the signal-to-noise ratio (SNR). Compared to [20], the cardinalities of the ED-PPM-ST codebooks in [20] are $\frac{(M-1)(M-2)}{2}$ and $\frac{(M-1)(M-2)(M-3)}{6}$ with two and three transmit antennas, respectively, and these cardinalities are exceeded by the cardinalities of proposed ST codebooks for all values of M.

The remainder of the paper is organized as follows. Section II introduces the system model. The structure of the ST encoder and decoder as well as the design criteria to be satisfied by the proposed construction are all highlighted in Section III. For the sake of clarity, a synopsis of the results is presented in Section IV where the ST codebooks are listed. In Section V, the design criteria are translated into a set of confinement rules to be respected by the ST codebooks in the cases of two and three transmit antennas. We next solve for the codebooks with maximum cardinalities satisfying the confinement rules for two and three transmit antennas in Section VI and Section VII, respectively. Finally, some numerical results are presented in Section VIII while Section IX concludes the paper.

II. System Model

Consider a M-ary PPM constellation for which the symbol duration is divided into M slots where a signal is transmitted in the m-th slot for the communication of the m-th symbol for $m \in \{1,\dots,M\} \triangleq \mathcal{S}$. The analyzed system corresponds to a MIMO IR-UWB system with P transmit antennas and Q receive antennas. The integration time will be denoted by T_i.

Analog energy detectors are considered where the decision variables are based on the energies collected in the different PPM slots. Assume that the antennas in the set $\mathcal{P} \subset \{1,\dots,P\}$ are pulsed simultaneously in a certain PPM slot, then the accumulated energy can be expressed as:

$$\mathfrak{e}_{\mathcal{P}} = \sum_{q=1}^{Q} \int_0^{T_i} \left(\sum_{p\in\mathcal{P}} r_{p,q}(t) + n_q(t) \right)^2 \mathrm{d}t, \tag{1}$$

where $r_{p,q}(t)$ stands for the end-to-end impulse response between the p-th transmit and q-th receive antennas comprising the effects of the UWB pulse shaper, the channel impulse response and the receiver filter. $n_q(t)$ is the filtered additive white Gaussian noise (AWGN) at the q-th antenna with zero mean, single-sided power spectral density N_0 and single-sided bandwidth W.

III. Space-Time Encoding/Decoding

A. Codeword Structure

The codewords are based on the structure that was proposed in [11] and [20] in the context of coherent and noncoherent systems, respectively. For $P=2$, the codewords are given by:

$$C(\mathbf{s}) = \begin{bmatrix} s_1 & \pi(s_2) \\ s_2 & s_1 \end{bmatrix}, \tag{2}$$

and, for $P=3$, the codewords are given by:

$$C(\mathbf{s}) = \begin{bmatrix} s_1 & \pi(s_3) & \pi(s_2) \\ s_2 & s_1 & \pi(s_3) \\ s_3 & s_2 & s_1 \end{bmatrix}. \tag{3}$$

In (2)-(3), $\mathbf{s} \triangleq (s_1, s_2)$ for $P=2$ and $\mathbf{s} \triangleq (s_1, s_2, s_3)$ for $P=3$. The cyclic permutation of order k is described by:

$$\pi^k(m) = (m+k-1) \mod M+1, \tag{4}$$

with $\pi^k(m_1, m_2, \dots) \triangleq (\pi^k(m_1), \pi^k(m_2), \dots)$ for simplicity of notation.

In (2) and (3), the (p,n)-th element of the codeword corresponds to the PPM slot that is occupied by the UWB pulse transmitted by the p-th transmit antenna during the n-th symbol duration. In this context, while the M-PPM constellation has M dimensions, we use the integers $s_1,\dots,s_P \in \{1,\dots,M\}$ to indicate the positions of the transmitted pulses.

The interest in maintaining the codeword structure given in (2)-(3) stems from the much desired property that the constructed codes will be shape-preserving with PPM since $\pi(s) \in \mathcal{S}$ whenever $s \in \mathcal{S}$. One can also identify the threaded structure of the codewords.

In the context of coherent communications, $\mathbf{s} \in \mathcal{S}^P$ where there are no constraints on the selection of the symbols $s_1,\dots,s_P$ from the M-PPM signal set [11]. On the other hand, for noncoherent systems, the diversity gain of the ST code will be lost if the entire set $\mathcal{S}^P$ is considered [20]. In this case, the values of the information vectors $\mathbf{s}$ must be confined to a convenient subset $\mathcal{C}$ of $\mathcal{S}^P$ in order to maintain the full diversity advantage as in the coherent case. The objective of this work is to solve for the set $\mathcal{C}$ having the maximum cardinality such that the association of the codewords in (2)-(3) with analog energy detectors will be capable of achieving a full transmit diversity gain while exceeding the rates of the codes proposed in [20].

B. Energy-Based Noncoherent Detection

The noncoherent Maximum-Likelihood (ML) detector decides in favor of the codeword that results in the combination of PPM slots that contain the maximum energy. The ML decision rule can be expressed under the following form:

$$\hat{\mathbf{s}} = \underset{\mathbf{s}\in\mathcal{C}}{\arg\max} \left\{ \sum_{n=1}^{N} E_n(\mathbf{s}) \right\}, \tag{5}$$

where $N = P$ stands for the block length and $\mathrm{E}_n(\mathbf{s})$ stands for the energy collected in the n-th symbol duration when the decoder is testing the assumption that the information vector is $\mathbf{s}$.

The energy $\mathrm{E}_n(\mathbf{s})$ is given by:

$$\mathrm{E}_n(\mathbf{s}) = \sum_{q=1}^{Q}\sum_{p=1}^{P} E^{(q)}_{n,C_{p,n}(\mathbf{s})}, \tag{6}$$

where $E^{(q)}_{n,m}$ stands for the energy collected at the q-th receive antenna in the m-th PPM slot of the n-th symbol duration. In (6), $C_{p,n}(\mathbf{s})$ stands for the (p,n)-th element of the codeword $C(\mathbf{s})$ given in (2) and (3).

Designate by $\mathcal{E} = \{E^{(q)}_{n,m}\ ;\ q = 1,\dots,Q\ ;\ n = 1,\dots,P\ ;\ m = 1,\dots,M\}$ the set comprising all QPM integrated energies at all receive antennas in the PPM slots of the P symbol durations. These energies need to be collected only once in the ST block duration in a way that is completely analogous to the detection of the non-coherent ST codes in [20]. In this context, the set $\mathcal{E}$ is evaluated independently from the tested codeword rendering the complexity of this part of the receiver independent from the rate of the code. In other words, at this level, the decoding complexity of the proposed scheme and of [20] are the same. Moreover, as in [20], the ML decoder can be completely implemented using adders thus contributing positively to the simplicity of the decoder. The impact of the codebook cardinality on the decoder complexity manifests in (5) where $|\mathcal{C}|$ possible combinations of elements of $\mathcal{E}$ need to be tested. In this context, the decoding complexity of this part increases linearly with the cardinality of the codebook necessitating the search among a larger number of elements for the proposed code as compared to [20]. For example, following from the cardinalities of the proposed code and of [20] (that have been reported in the introduction section), (5) involves testing $\frac{M}{M-1}$ more elements compared to [20] in the case of two transmit antennas where this ratio decreases with M.

C. Design Criteria

From (2) and (3), since we are keeping the same codeword structure as in [20], the presented noncoherent ED-PPM-ST code construction will be based on the design criteria in [20]. These criteria are based on minimizing the pairwise error probability when transmitting the information vector $\mathbf{s}$ and deciding in favor of $\mathbf{s}' \neq \mathbf{s}$. In particular, the two following criteria need to be respected by the codebook $\mathcal{C}$ from which the information vectors are to be carved [1].

[1]It is worth noting that these design criteria constitute a set of sufficient but not necessary conditions for achieving full diversity with energy detectors and PPM.

1) Criterion 1: At most one transmit antenna can be pulsed during each PPM time slot.

2) Criterion 2: The energies of the N pulses received from any of the P antennas must not be all included in the decision variable $\sum_{n=1}^{N} E_n(\mathbf{s}')$ for any $\mathbf{s}' \neq \mathbf{s}$. If this is not the case, then when comparing $\sum_{n=1}^{N} E_n(\mathbf{s}')$ with $\sum_{n=1}^{N} E_n(\mathbf{s})$ (for the sake of ML detection), the energy received from a specific antenna will be common to the two decision variables and, hence, will simplify out (for high SNRs). Therefore, transmissions from this antenna will have no impact on the ML decoder implying a reduced diversity order.

Criterion 2 has been mathematically formulated in [20]. An equivalent formulation that simplifies the code design will be presented in what follows. Consider the n-th symbol duration. The p-th antenna is transmitting in the PPM slot $C_{p,n}(\mathbf{s})$ while, from (6), the ML decoder is adding the energies in the slots $C_{1,n}(\mathbf{s}'),\dots,C_{P,n}(\mathbf{s}')$ when testing the assumptions that $\mathbf{s}'$ was transmitted. Based on this observation, we define $I_{n,p}$ as:

$$I_{n,p} = \begin{cases} 1, & \exists\, p' \in \{1,\dots,P\} \mid C_{p',n}(\mathbf{s}') = C_{p,n}(\mathbf{s}); \\ 0, & \text{otherwise.} \end{cases}, \tag{7}$$

where $I_{n,p} = 1$ means that the energy received from the p-th transmit antenna is included in the total energy collected in the n-th symbol duration $\mathrm{E}_n(\mathbf{s}')$.

Based on the notation in (7), criterion 2 can be expressed as a set of P conditions as follows:

$$(I_{1,p},\dots,I_{N,p}) \neq (1,\dots,1)\ \ ;\ \ p = 1,\dots,P. \tag{8}$$

While the error probability decays exponentially with the SNR [22] for the considered UWB channel model [23], criterion 2 is used to guarantee the full diversity advantage based on the performance analysis presented in [20]. In this context, the SER of the ST scheme can be written as the product of P terms each related exclusively to a specific transmit antenna. If (8) is not satisfied for a certain $p \in \{1,\dots,P\}$, the corresponding multiplicative term will disappear from the SER expression implying that the P-fold decrease in the SER can not be guaranteed resulting in a reduced diversity gain. Interested readers are referred to [20] (eq. (32) and the subsequent analysis) for more details on the relation between criterion 2 and the achievable diversity gains.

IV. Synopsis of the Results

For the sake of clarity and since solving for the codebook $\mathcal{C}$ with maximum cardinality involves numerous preliminary stages, a synopsis of the results will be provided in this section.

The analysis provided in sections V-A and VI shows that the codebook with maximum cardinality in the case of two transmit antennas can be constructed as follows (M even):

$$\mathcal{C} = \{\pi^m(1,n)\ ;\ n = 2,\dots,M/2\ ;\ m = 0,\dots,M-1\}, \tag{9}$$

that attains the maximum cardinality of $\frac{M(M-2)}{2}$.

Similarly, the analysis provided in sections V-B and VII is culminated by the following solution of the codebook with

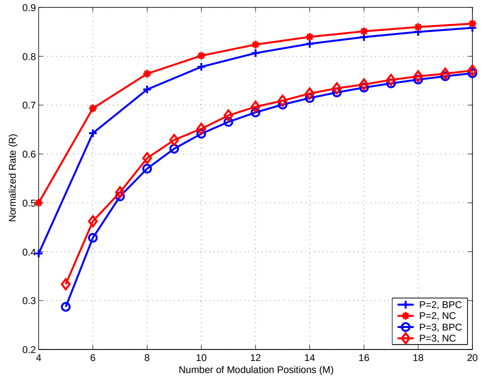

Fig. 1. Normalized rate of the new-codes (NC) versus the best-previous-codes (BPC) in [20].

maximum cardinality in the case of three transmit antennas:

$$\begin{aligned} \mathcal{C} = \{&\pi^m\left(1, \pi(1), \pi^n(1)\right) \ ; \\ & n = 2, \dots, M-3 \ ; \ m = 0, \dots, M-1\} \ \cup \\ &\{\pi^m\left(1, \pi^{n_1}(1), \pi^{n_2}(1)\right) \ ; \ n_1 = 2, \dots, k-1 \ ; \\ & n_2 = 2n_1, \dots, M-2-n_1 \ ; \ m = 0, \dots, M-1\}, \end{aligned} \quad (10)$$

achieving the cardinality of $M\left[M-4+\left\lfloor\frac{(M-5)(M-6)}{6}\right\rfloor\right]$. The integer k is obtained by writing M under the form $M = 3k+k'$ with $k' \in \{-1, 0, 1\}$. The analysis in sections V, VI and VII also reveals the existence of a number of equivalent codebooks that have the same cardinalities as the codebooks given in (9) and (10). In this context, these equivalent codes achieve the same performance as (9)-(10) since the respected construction constraints are the same.

The normalized rate of a ST code is defined as $R = \frac{\log_2(|\mathcal{C}|)}{P \log_2(M)}$ where $\log_2(M)$ is the number of bits for a single-antenna system with M-PPM. The proposed codes are to be compared with the codes in [20] that satisfy the same construction constraints. Fig. 1 compares the normalized rates of the new-codes (NC) and the best-pervious-codes (BPC) in [20] for which $|\mathcal{C}| = \binom{M-1}{P}$. Results show that the proposed codes have higher rates for all values of M.

Unlike the determinant criterion [21] that is adopted for the design of coherent ST codes where the minimum determinant can assume any positive value, the condition in (8) does not manifest such continuous variations in the sense that it is either satisfied or not. Therefore, the coherent-QAM and noncoherent-PPM ST code design problems can be perceived as dual problems where, in the first case, the minimum determinant is maximized for a fixed rate while, in the second case, the rate is maximized for a fixed quality-assurance indicator (criterion 2). In this context, for all codes satisfying criterion 2 (along with the other design constraints such as ED, shape-preserving constraint,...), the SER performance will enhance with the cardinality of the codebook since more information bits will be included in each codeword without jeopardizing any of the desired features of the code. This is especially true since, for the sake of comparing ST codes with different rates, the SER must be plotted as a function of the SNR per information bit that turns to the good advantage of high-rate codes. This justifies the enhanced performance levels compared to [20] where exactly the same set of construction constraints is satisfied while inserting more information bits per codeword. Based on the above observation, the association of the codewords with any subset of $\mathcal{C}$ (in (9) or (10)) will result, evidently, in reduced SERs.

V. Confinement of the Set $\mathcal{S}^P$

A. $P = 2$ Transmit Antennas

1) First Confinement: Respecting criterion 1 by refraining the two transmit antennas from transmitting in the same PPM slot results in the following relations following from (2):

$$s_1 \neq s_2 \quad \text{and} \quad s_1 \neq \pi(s_2). \quad (11)$$

2) Second Confinement: The second confinement follows from satisfying criterion 2. We start by defining the following logical statements:

$$\begin{aligned} &l_1 : s'_1 = s_1 \quad ; \quad l_2 : s'_1 = s_2 \quad ; \quad l_3 : s'_1 = \pi(s_2) \\ &l_4 : s'_2 = s_1 \quad ; \quad l_5 : s'_2 = s_2 \quad ; \quad l_6 : \pi(s'_2) = s_1. \end{aligned} \quad (12)$$

From (2), the first antenna is transmitting in slot s_1 of the first symbol duration and in slot $\pi(s_2)$ of the second symbol duration. Based on (5), the ML decoder is collecting the energy in slots s'_1 and s'_2 of the first symbol duration and in slots s'_1 and $\pi(s'_2)$ of the second symbol duration. Consequently, referring to (7), $I_{1,1} = 1$ if $s'_1 = s_1$ or $s'_2 = s_1$ while $I_{2,1} = 1$ if $s'_1 = \pi(s_2)$ or $\pi(s'_2) = \pi(s_2)$. Using the logical statements in (12), $I_{1,1} = 1 \ \Leftrightarrow \ l_1 \vee l_4$ and $I_{2,1} = 1 \ \Leftrightarrow \ l_3 \vee l_5$ where $\vee$ stands for the logical *or* operator. Therefore, the condition in (8) will not be satisfied for $p = 1$ when:

$$(I_{1,1}, I_{2,1}) = (1, 1) \ \Leftrightarrow \ (l_1 \vee l_4) \wedge (l_3 \vee l_5), \quad (13)$$

where $\wedge$ stands for the logical *and* operator.

On the other hand, the second antenna is transmitting in slots s_2 and s_1 of the first and second symbol durations, respectively. Consequently, $I_{1,2} = 1$ if $s'_1 = s_2$ or $s'_2 = s_2$ while $I_{2,2} = 1$ if $s'_1 = s_1$ or $\pi(s'_2) = s_1$ following from (7). Therefore, using the logical statements in (12), the condition in (8) will not be satisfied for $p = 2$ when:

$$(I_{1,2}, I_{2,2}) = (1, 1) \ \Leftrightarrow \ (l_2 \vee l_5) \wedge (l_1 \vee l_6). \quad (14)$$

Keeping in mind that criterion 2 follows from the pairwise error probability of transmitting $\mathbf{s}$ and deciding in favor of $\mathbf{s}'$, the following proposition holds.

Proposition *1:* Criterion 2 is satisfied if $(s'_1, s'_2) \neq (s_2, \pi^{-1}(s_1))$ and $(s'_1, s'_2) \neq (\pi(s_2), s_1)$. In other words, for every element (s_1, s_2) of $\mathcal{C}$, the two elements $(s_2, \pi^{-1}(s_1))$ and $(\pi(s_2), s_1)$ must not belong to $\mathcal{C}$.

Proof: From (13), $(l_1 \vee l_4) \wedge (l_3 \vee l_5)$ is true if either one of the following logical statements is true: $l_1 \wedge l_3$ or $l_1 \wedge l_5$ or $l_4 \wedge l_3$ or $l_4 \wedge l_5$. (i): $l_1 \wedge l_3$ is always false since it implies that $s'_1 = s_1 = \pi(s_2)$ where the second equality contradicts (11). (ii): $l_1 \wedge l_5$ results in $(s'_1, s'_2) = (s_1, s_2)$ which is not possible since the pairwise error probability is evaluated between distinct pairs of symbols. (iii): $l_4 \wedge l_5 \Leftrightarrow s'_2 = s_1 = s_2$ which is a false statement since $s_1 \neq s_2$ from (11). Therefore, the logical statement in (13) is true if $l_4 \wedge l_3$ is true which is equivalent to $s'_1 = \pi(s_2)$ and $s'_2 = s_1$.

Similarly, from (14), $(l_2 \vee l_5) \wedge (l_1 \vee l_6) = (l_2 \wedge l_1) \vee (l_2 \wedge l_6) \vee (l_5 \wedge l_1) \vee (l_5 \wedge l_6)$. (i): $l_2 \wedge l_1 \Leftrightarrow s'_1 = s_1 = s_2$ contradicting (11). (ii): $l_5 \wedge l_6 \Leftrightarrow s_1 = \pi(s'_2) = \pi(s_2)$ contradicting (11). (iii): $l_5 \wedge l_1 \Leftrightarrow (s'_1, s'_2) = (s_1, s_2)$ which is false since $\mathbf{s}' \neq \mathbf{s}$. Consequently, $(l_2 \vee l_5) \wedge (l_1 \vee l_6) \Leftrightarrow (l_2 \wedge l_6) \Leftrightarrow (s'_1, \pi(s'_2)) = (s_2, s_1) \Leftrightarrow (s'_1, s'_2) = (s_2, \pi^{-1}(s_1))$ completing the proof. ■

As a conclusion, from (11) and proposition 1, the codebook $\mathcal{C}$ to be associated with the codewords in (2) in the case of two transmit antennas must satisfy:

$$\mathcal{C} = \big\{(s_1, s_2) \in \mathcal{S}^2 \mid s_2 \neq s_1 \ , \ \pi(s_2) \neq s_1 \ , \\ (s_1, s_2)^* \notin \mathcal{C} \ , \ (s_1, s_2)^{**} \notin \mathcal{C}\big\} \quad (15)$$

where $(s_1, s_2)^* \triangleq (s_2, \pi^{-1}(s_1))$ and $(s_1, s_2)^{**} \triangleq (\pi(s_2), s_1)$ will be denoted by the conjugates of (s_1, s_2) that are distinct.

B. $P = 3$ Transmit Antennas

1) First Confinement: From (3), criterion 1 is satisfied if the following conditions hold:

$$s_1 \neq s_2 \quad ; \quad s_1 \neq s_3 \quad ; \quad s_2 \neq s_3 \quad ; \\ s_1 \neq \pi(s_3) \ ; \ s_2 \neq \pi(s_3) \ ; \ s_1 \neq \pi(s_2), \qquad (16)$$

where the elements of each column of a codeword need to be distinct.

2) Second Confinement: We define the following additional logical statements that complement those defined in (12) for the case $P = 3$:

$$l_7 : s'_3 = s_1 \quad ; \quad l_8 : s'_3 = s_3 \ ; \ l_9 : s'_1 = \pi(s_3) \ ; \\ l_{10} : s'_2 = \pi(s_3) \ ; \ l_{11} : s'_3 = s_2 \ ; \ l_{12} : \pi(s'_3) = s_1 \ ; \\ l_{13} : s'_2 = s_3 \quad ; \quad l_{14} : s'_1 = s_3 \ ; \ l_{15} : \pi(s'_3) = s_2. \quad (17)$$

Proposition *2:* Criterion 2 is satisfied if for every element (s_1, s_2, s_3) of $\mathcal{C}$, the following elements do not belong to $\mathcal{C}$:

$$\mathbf{s}^{(1)} \triangleq (\pi(s_3), s_1, s_2) \ ; \ \mathbf{s}^{(2)} \triangleq (\pi(s_2), \pi(s_3), s_1)$$
$$\begin{cases} \mathbf{s}^{(3)} \triangleq (\pi(s_3), s_2, s_1) \\ \mathbf{s}^{(4)} \triangleq (s_2, s_1, s_3) \\ \mathbf{s}^{(5)} \triangleq (s_1, s_3, \pi^{-1}(s_2)) \end{cases} ; \quad \text{if} \ \ s_2 \neq \pi(s_1)$$
$$\begin{cases} \mathbf{s}^{(6)} \triangleq (\pi(s_2), s_1, s_3) \\ \mathbf{s}^{(7)} \triangleq (s_1, s_3, s_2) \\ \mathbf{s}^{(8)} \triangleq (s_3, s_2, \pi^{-1}(s_1)) \end{cases} ; \quad \text{if} \ \ s_3 \neq \pi(s_2)$$
$$\begin{cases} \mathbf{s}^{(9)} \triangleq (s_1, \pi(s_3), s_2) \\ \mathbf{s}^{(10)} \triangleq (\pi(s_3), s_2, \pi^{-1}(s_1)) \\ \mathbf{s}^{(11)} \triangleq (s_2, \pi^{-1}(s_1), s_3) \end{cases} ; \quad \text{if} \ \ s_1 \neq \pi^2(s_3). \quad (18)$$

where the elements $\mathbf{s}^{(1)}, \dots, \mathbf{s}^{(11)}$ will be referred to as the conjugates of $\mathbf{s}$.

Proof: From (8), (7), (12) and (17), $(I_{1,1}, I_{2,1}, I_{3,1}) = (1, 1, 1) \Leftrightarrow (l_1 \vee l_4 \vee l_7) \wedge (l_8 \vee l_9 \vee l_{10}) \wedge (l_3 \vee l_5 \vee l_{11})$, $(I_{1,2}, I_{2,2}, I_{3,2}) = (1, 1, 1) \Leftrightarrow (l_2 \vee l_5 \vee l_{11}) \wedge (l_1 \vee l_4 \vee l_{12}) \wedge (l_8 \vee l_9 \vee l_{13})$ and $(I_{1,3}, I_{2,3}, I_{3,3}) = (1, 1, 1) \Leftrightarrow (l_8 \vee l_{13} \vee l_{14}) \wedge (l_2 \vee l_5 \vee l_{15}) \wedge (l_1 \vee l_6 \vee l_{12})$. Simplifying the resulting logical statements while observing that some of them are always false since they violate (16) (or $\mathbf{s}' = \mathbf{s}$) results in $\mathbf{s}' = \mathbf{s}^{(i)}$ for $i = 1, \dots, 11$. The derivations are straightforward yet long and tedious and, hence, will be omitted for the sake of brevity. A sketch of the proof and the derivations of some conjugates are provided in Appendix A. These derivations are analogous to those provided for the proof of proposition 1. ■

Among the conjugates of $\mathbf{s}$, the conjugates $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$ are unconditional conjugates that always exist for any information vector $\mathbf{s}$ satisfying (16) while the conjugates $\mathbf{s}^{(3)}, \dots, \mathbf{s}^{(11)}$ are conditional conjugates that exist only if the components of $\mathbf{s}$ satisfy certain conditions. These conditions follow from the fact that the conjugates must satisfy (16) so that, in their turn, they will be included in $\mathcal{C}$. In fact, if these conditions are not satisfied, then the corresponding conjugates will not belong to $\mathcal{C}$ and, hence, will not affect the fulfillment of criterion 2.

As a conclusion, from (16) and proposition 2, the codebook $\mathcal{C}$ to be associated with the codewords in (3) for three transmit antennas must be constructed as follows:

$$\mathcal{C} = \Big\{(s_1, s_2, s_3) \in \mathcal{S}^3 \mid s_1 \neq s_2 \ , \ s_1 \neq s_3 \ , \ s_2 \neq s_3 \ , \\ s_1 \neq \pi(s_3) \ , \ s_2 \neq \pi(s_3) \ , \ s_1 \neq \pi(s_2) \\ ; \ \mathbf{s}^{(1)}, \dots, \mathbf{s}^{(11)} \notin \mathcal{C}\Big\}, \quad (19)$$

where $\mathbf{s}^{(1)}, \dots, \mathbf{s}^{(11)}$ are defined in (18).

VI. Solving for $\mathcal{C}$ with 2 Transmit Antennas

In the case $P = 2$, we will next solve for the set $\mathcal{C}$ whose structure is given (15) such that this set has the the maximum cardinality.

Proposition *3:* The maximum possible cardinality of the codebook $\mathcal{C}$ whose structure satisfies (15) is $\frac{M(M-2)}{2}$.

Proof: We will prove this proposition by contradiction. Denote by $\bar{\mathcal{C}}$ the set containing the conjugates of the elements of $\mathcal{C}$. In this case, $|\mathcal{C}| + |\bar{\mathcal{C}}| \leq M(M-2)$ since every element s_2 of $\mathcal{S}$ can be associated with only $M - 2$ elements s_1 of the same set such that the conditions in (11) are satisfied (note that $(s_1, s_2)^*$ and $(s_1, s_2)^{**}$ will satisfy (11) when (s_1, s_2) satisfy this equation). Assume that $|\bar{\mathcal{C}}| < |\mathcal{C}|$, then at least two distinct elements (s_1, s_2) and (s'_1, s'_2) of $\mathcal{C}$ will share either the same first conjugate or the same second conjugate. Now the relation $(s_1, s_2)^* = (s'_1, s'_2)^*$ will imply that $(s_1, s_2) = (s'_1, s'_2)$ since the function $\pi^{-1}(.)$ is bijective and the relation $(s_1, s_2)^{**} = (s'_1, s'_2)^{**}$ will imply that $(s_1, s_2) = (s'_1, s'_2)$ as well since the function $\pi(.)$ is bijective. This will contradict the fact that (s_1, s_2) and (s'_1, s'_2) are distinct implying that the assumption can not hold and $|\bar{\mathcal{C}}| \geq |\mathcal{C}|$. Combining this inequality with $|\mathcal{C}| + |\bar{\mathcal{C}}| \leq M(M-2)$ results in $|\mathcal{C}| \leq \frac{M(M-2)}{2}$. ■

We will assume that M is even in what follows.

Proposition *4:* The subset of $\mathcal{S}^2$ whose $M(M-2)$ elements satisfy (11) can be written under the following form:

$$\mathcal{S}_0 = \{\pi^m(1, n), \pi^m(n, M) \ ; \\ n = 2, \dots, M/2 \ ; \ m = 0, \dots, M-1\}. \quad (20)$$

Proof: The proof is provided in Appendix B. ■

Lemma *1:* From propositions 3 and 4, the codebook $\mathcal{C}$ with maximum cardinality can be constructed as follows:

$$\mathcal{C} = \{\pi^m(1, n) \ ; \ n = 2, \dots, M/2 \ ; \ m = 0, \dots, M-1\}, \quad (21)$$

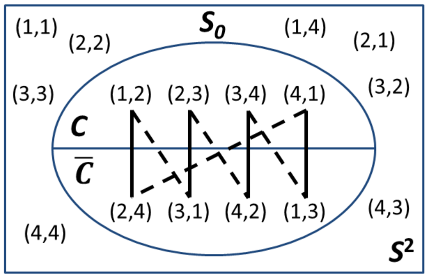

Fig. 2. Partitioning of $\mathcal{S}^2$ for $M = 4$. Solid and dashed lines correspond to the first and second conjugates, respectively.

which corresponds to the solution provided in (9).

Proof: We will prove that $\bar{\mathcal{C}} = \{\pi^m(n, M)\ ;\ n = 2 \cdots M/2\ ;\ m = 0 \cdots M-1\}$. In this case, the set $\mathcal{S}_0$ in (20) is partitioned into the two subsets $\mathcal{C}$ and $\bar{\mathcal{C}}$ (of equal cardinality $\frac{M(M-2)}{2}$ each) where the first set contains the information vectors to be associated with (2) while the second set contains the conjugates of these vectors so that (15) is respected.

Let $(s_1, s_2) = \pi^m(1, n)$: $(s_1, s_2)^* = (\pi^m(n), \pi^{m-1}(1)) = \pi^m(n, M)$. Similarly, $(s_1, s_2)^{**} = (\pi^{m+1}(n), \pi^m(1)) = \pi^{m+1}(n, M)$. Consequently, for a given value of n, the conjugates of the M elements of $\mathcal{C}_n \triangleq \{\pi^m(1, n)\}_{m=0}^{M-1}$ are: $\{\pi^m(n, M)\}_{m=0}^{M-1} \cup \{\pi^{m+1}(n, M)\}_{m=0}^{M-1} = \{\pi^m(n, M)\}_{m=0}^{M-1} \cup \{\pi^m(n, M)\}_{m=1}^{M} = \{\pi^m(n, M)\}_{m=0}^{M-1} \triangleq \bar{\mathcal{C}}_n$ since $\pi^M(s) = \pi^0(s) = s$. Therefore, all the conjugates of the elements of $\mathcal{C} = \bigcup_{n=2}^{\frac{M}{2}} \mathcal{C}_n$ are confined in $\bar{\mathcal{C}} = \bigcup_{n=2}^{\frac{M}{2}} \bar{\mathcal{C}}_n$. ∎

It is worth noting that $\bar{\mathcal{C}}$ (rather than $\mathcal{C}$) can be selected as the codebook. In this context, $\mathcal{C}$ and $\bar{\mathcal{C}}$ constitute two equivalent codebooks that have the same cardinality.

Finally, while $\mathbf{s}'^* \neq \mathbf{s}^*$ and $\mathbf{s}'^{**} \neq \mathbf{s}^{**}$ for $\mathbf{s}' \neq \mathbf{s}$, the relation $\mathbf{s}'^{**} = \mathbf{s}^*$ can hold. Consider the elements $\pi^m(1, n)$ and $\pi^{m+1}(1, n)$ of $\mathcal{C}_n$, it follows directly that $[\pi^{m+1}(1, n)]^* = [\pi^m(1, n)]^{**}$ and, hence, $\mathcal{C}_n$ possesses the following important property in the interest of maximizing the cardinality of $\mathcal{C}$: the $2M$ conjugates of elements of $\mathcal{C}_n$ correspond in reality to only M distinct elements where each one of these elements is concurrently the first conjugate of one element of $\mathcal{C}_n$ and the second conjugate of another element of $\mathcal{C}_n$. In other words, while $|\mathcal{C}_n| = M$, $\bar{\mathcal{C}}_n$ has the minimum possible cardinality of M since a smaller cardinality will imply that at least one element of $\bar{\mathcal{C}}_n$ is concurrently the first (or second) conjugate of two or more elements of $\mathcal{C}_n$ which is not possible. The partitioning of $\mathcal{S}^2$ is better illustrated in Fig. 2 for $M = 4$.

VII. Solving for $\mathcal{C}$ with 3 Transmit Antennas

We next consider the case $P = 3$ and we solve for the codebook $\mathcal{C}$ with maximum cardinality such that its structure satisfies (19). Denote by $\mathcal{S}_0$ the subset of $\mathcal{S}^3$ whose elements satisfy the conditions in (16) that follow from respecting criterion 1. Through regular counting techniques, it can be proven that $\mathcal{S}_0$ comprises $M(M-3)^2$ elements.

A. Structure of $\mathcal{C}$

Proposition 5: The codebook $\mathcal{C} \subset \mathcal{S}_0$ with maximum cardinality satisfies the following:

$$\{\pi^m(\mathbf{s})\ ;\ m = 0, \ldots, M-1\} \subset \mathcal{C} \quad \forall \quad \mathbf{s} \in \mathcal{C}, \tag{22}$$

implying that all permutations of elements of $\mathcal{C}$ are also included in $\mathcal{C}$.

Proof: The proof is provided in Appendix C. ∎

Proposition 5 suggests that if a certain codebook does not satisfy the structure in (22) (as is the case of the codebook in [20]), then it is always possible to find another codebook comprising a larger number of elements and, hence, the former codebook should be disregarded. Based on proposition 5 and (19), the structure of the codebook respecting the two design criteria while having the maximum cardinality can be simplified as follows:

$$\begin{aligned}\mathcal{C} = \Big\{&\pi^m(1, \pi^{n_1}(1), \pi^{n_2}(1)) \in \mathcal{S}^3 \mid n_1 \not\equiv 0\ ,\ n_1 \not\equiv -1\\ &,\ n_2 \not\equiv 0\ ,\ n_2 \not\equiv -1\ ,\ n_2 \not\equiv n_1\ ,\ n_1 \not\equiv n_2 + 1\\ &;\ m = 0, \ldots, M-1\ ;\ \mathbf{s}^{(1)}, \ldots, \mathbf{s}^{(11)} \notin \mathcal{C}\Big\},\end{aligned} \tag{23}$$

where $\equiv$ and $\not\equiv$ stand for the equality and inequality modulo M, respectively. In the same way, for two elements $\mathbf{s}$ and $\mathbf{s}'$ of $\mathcal{S}^3$, the relation $\mathbf{s} \equiv \mathbf{s}'$ (resp. $\mathbf{s} \not\equiv \mathbf{s}'$) means that there exists (resp. does not exist) an integer m such that $\mathbf{s} = \pi^m(\mathbf{s}')$.

Finally, proposition 5 simplifies the design problem from solving for three unknowns as in (19) to solving for two unknowns as in (23). Based on the introduced modulo-M equivalence between the elements of $\mathcal{S}^3$, the conjugates in (18) can be rewritten as follows:

$$\begin{aligned}
&\mathbf{s}^{(1)} = \pi^{m+n_2+1}(1, \pi^{-n_2-1}(1), \pi^{n_1-n_2-1}(1))\\
&\mathbf{s}^{(2)} = \pi^{m+n_1+1}(1, \pi^{n_2-n_1}(1), \pi^{-n_1-1}(1))\\
&\begin{cases}\mathbf{s}^{(3)} = \pi^{m+n_2+1}(1, \pi^{n_1-n_2-1}(1), \pi^{-n_2-1}(1))\\ \mathbf{s}^{(4)} = \pi^{m+n_1}(1, \pi^{-n_1}(1), \pi^{n_2-n_1}(1))\\ \mathbf{s}^{(5)} = \pi^{m}(1, \pi^{n_2}(1), \pi^{n_1-1}(1))\end{cases}\\
&\text{if} \quad n_1 \not\equiv 1\\
&\begin{cases}\mathbf{s}^{(6)} = \pi^{m+n_1+1}(1, \pi^{-n_1-1}(1), \pi^{n_2-n_1-1}(1))\\ \mathbf{s}^{(7)} = \pi^{m}(1, \pi^{n_2}(1), \pi^{n_1}(1))\\ \mathbf{s}^{(8)} = \pi^{m+n_2}(1, \pi^{n_1-n_2}(1), \pi^{-n_2-1}(1))\end{cases}\\
&\text{if} \quad n_2 \not\equiv n_1 + 1\\
&\begin{cases}\mathbf{s}^{(9)} = \pi^{m}(1, \pi^{n_2+1}(1), \pi^{n_1}(1))\\ \mathbf{s}^{(10)} = \pi^{m+n_2+1}(1, \pi^{n_1-n_2-1}(1), \pi^{-n_2-2}(1))\\ \mathbf{s}^{(11)} = \pi^{m+n_1}(1, \pi^{-n_1-1}(1), \pi^{n_2-n_1}(1))\end{cases}\\
&\text{if} \quad n_2 \not\equiv -2.
\end{aligned} \tag{24}$$

It can be observed that the conjugates $\mathbf{s}^{(1)}, \ldots, \mathbf{s}^{(11)}$ belong to $\mathcal{S}_0$ whenever $\mathbf{s}$ belongs to $\mathcal{S}_0$. Moreover, in a way similar to observation 8 in Appendix C, it is straightforward to prove that the conjugates in (24) satisfy the following relations for any element $\mathbf{s}$ of $\mathcal{S}_0$:

$$\left[\mathbf{s}^{(1)}\right]^{(1)} \equiv \mathbf{s}^{(2)} \quad ; \quad \left[\mathbf{s}^{(2)}\right]^{(2)} \equiv \mathbf{s}^{(1)}, \tag{25}$$

$$\begin{aligned}&\left[\mathbf{s}^{(i)}\right]^{(\bar{i})} \equiv \left[\mathbf{s}^{(\bar{i})}\right]^{(i)} \equiv \mathbf{s} \quad \text{for} \quad (i, \bar{i}) =\\ &(1,2), (3,8), (4,4), (5,9), (6,11), (7,7), (10,10).\end{aligned} \tag{26}$$

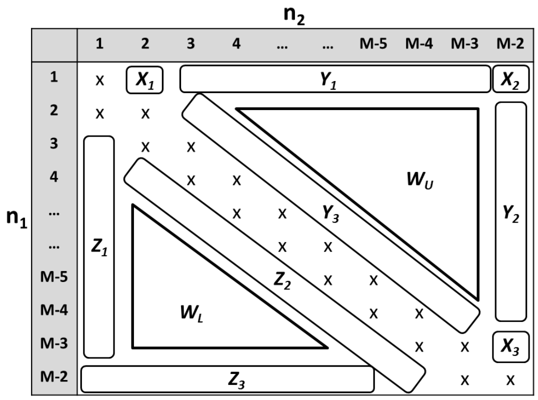

Fig. 3. Partitioning of $\mathcal{S}_0$. 'x' denotes the symbols that do not satisfy criterion 1.

B. Partitioning of $\mathcal{S}_0$

An extensive investigation of the properties of the conjugates in (24) revealed that the search for the codebook with maximum cardinality can be significantly simplified if the set $\mathcal{S}_0$ is partitioned as follows:

$$\mathcal{S}_0 = (\underbrace{\mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{X}_3}_{\triangleq \mathcal{X}}) \cup (\underbrace{\mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{Y}_3}_{\triangleq \mathcal{Y}}) \cup (\underbrace{\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3}_{\triangleq \mathcal{Z}}) \cup (\underbrace{\mathcal{W}_U \cup \mathcal{W}_L}_{\triangleq \mathcal{W}}), \quad (27)$$

where this partitioning is better depicted in Fig. 3.

Elements of the same subset of the partition manifest similar properties as will be explained later. From (24), we define the logical statements E_1, E_2 and E_3 as follows:

$$E_1 : n_1 \not\equiv 1 \quad ; \quad E_2 : n_2 \not\equiv n_1 + 1 \quad ; \quad E_3 : n_2 \not\equiv -2, \quad (28)$$

where $\overline{E_1} \wedge \overline{E_2} \wedge \overline{E_3}$ is always false.

1) The set $\mathcal{X}$: For the elements of $\mathcal{X}$, only one of the statements in (28) holds and, consequently, each one of these elements possesses five conjugates. Elements of $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_3$ satisfy $\overline{E_1} \wedge \overline{E_2} \wedge E_3$, $\overline{E_1} \wedge E_2 \wedge \overline{E_3}$ and $E_1 \wedge \overline{E_2} \wedge \overline{E_3}$, respectively, resulting in:

$$\begin{aligned}
\mathcal{X}_1 &= \{\pi^m(1, \pi(1), \pi^2(1)) \ ; \ m = 0, \dots, M-1\} \\
\mathcal{X}_2 &= \{\pi^m(1, \pi(1), \pi^{-2}(1)) \ ; \ m = 0, \dots, M-1\} \\
\mathcal{X}_3 &= \{\pi^m(1, \pi^{-3}(1), \pi^{-2}(1)) \ ; \ m = 0, \dots, M-1\}, \quad (29)
\end{aligned}$$

where each one of these sets contains M elements. In addition to the two conjugates $\{\mathbf{s}^{(1)}, \mathbf{s}^{(2)}\}$, elements of $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_3$ possess the three additional conjugates $\{\mathbf{s}^{(9)}, \mathbf{s}^{(10)}, \mathbf{s}^{(11)}\}$, $\{\mathbf{s}^{(6)}, \mathbf{s}^{(7)}, \mathbf{s}^{(8)}\}$ and $\{\mathbf{s}^{(3)}, \mathbf{s}^{(4)}, \mathbf{s}^{(5)}\}$, respectively.

2) The set $\mathcal{Y}$: For the elements of $\mathcal{Y}$, two of the statements in (28) hold and, consequently, each one of these elements possesses eight conjugates. Elements of $\mathcal{Y}_1$, $\mathcal{Y}_2$ and $\mathcal{Y}_3$ satisfy $\overline{E_1} \wedge E_2 \wedge E_3$, $E_1 \wedge E_2 \wedge \overline{E_3}$ and $E_1 \wedge \overline{E_2} \wedge E_3$, respectively, resulting in:

$$\begin{aligned}
\mathcal{Y}_1 &= \{\pi^m(1, \pi(1), \pi^n(1)) \ ; \\
&\qquad n = 3, \dots, M-3 \ ; \ m = 0, \dots, M-1\} \\
\mathcal{Y}_2 &= \{\pi^m(1, \pi^n(1), \pi^{-2}(1)) \ ; \\
&\qquad n = 2, \dots, M-4 \ ; \ m = 0, \dots, M-1\} \\
\mathcal{Y}_3 &= \{\pi^m(1, \pi^n(1), \pi^{n+1}(1)) \ ; \\
&\qquad n = 2, \dots, M-4 \ ; \ m = 0, \dots, M-1\}, \quad (30)
\end{aligned}$$

where each one of these sets contains $M(M-5)$ elements. Elements of $\mathcal{Y}_1$, $\mathcal{Y}_2$ and $\mathcal{Y}_3$ possess all conjugates except $\{\mathbf{s}^{(3)}, \mathbf{s}^{(4)}, \mathbf{s}^{(5)}\}$, $\{\mathbf{s}^{(9)}, \mathbf{s}^{(10)}, \mathbf{s}^{(11)}\}$ and $\{\mathbf{s}^{(6)}, \mathbf{s}^{(7)}, \mathbf{s}^{(8)}\}$, respectively.

3) The set $\mathcal{Z} \cup \mathcal{W}$: In this case, all statements in (28) hold and elements of $\mathcal{Z} \cup \mathcal{W}$ possess eleven conjugates. For convenience, the set $\mathcal{Z}$ will be further partitioned into three sets as follows:

$$\begin{aligned}
\mathcal{Z}_1 &= \{\pi^m(1, \pi^n(1), \pi^1(1)) \ ; \\
&\qquad n = 3, \dots, M-3 \ ; \ m = 0, \dots, M-1\} \\
\mathcal{Z}_2 &= \{\pi^m(1, \pi^n(1), \pi^{n-2}(1)) \ ; \\
&\qquad n = 4, \dots, M-2 \ ; \ m = 0, \dots, M-1\} \\
\mathcal{Z}_3 &= \{\pi^m(1, \pi^{-2}(1), \pi^n(1)) \ ; \\
&\qquad n = 1, \dots, M-5 \ ; \ m = 0, \dots, M-1\}, \quad (31)
\end{aligned}$$

where each one of these sets contains $M(M-5)$ elements.

The set $\mathcal{W}$ will be further partitioned into two sets as follows ($m = 0, \dots, M-1$):

$$\begin{aligned}
\mathcal{W}_U &= \{\pi^m(1, \pi^{n_1}(1), \pi^{n_2}(1)) \ ; \ n_1 = 2, \dots, M-5 \ ; \\
&\quad n_2 = 4, \dots, M-3 \mid n_2 - n_1 \in \{2, \dots, M-5\}\} \\
\mathcal{W}_L &= \{\pi^m(1, \pi^{n_1}(1), \pi^{n_2}(1)) \ ; \ n_1 = 5, \dots, M-3 \ ; \\
&\quad n_2 = 2, \dots, M-6 \mid n_1 - n_2 \in \{3, \dots, M-5\}\}. \quad (32)
\end{aligned}$$

C. Partitioning of $\mathcal{C}$

Based on the partitioning of $\mathcal{S}_0$ according to (27), the codebook $\mathcal{C}$ in its turn will be partitioned as $\mathcal{C} = \mathcal{C}_{\mathcal{X}} \cup \mathcal{C}_{\mathcal{Y} \cup \mathcal{Z}} \cup \mathcal{C}_{\mathcal{W}}$ where $\mathcal{C}_{\mathcal{X}} \subset \mathcal{X}$, $\mathcal{C}_{\mathcal{Y} \cup \mathcal{Z}} \subset (\mathcal{Y} \cup \mathcal{Z})$ and $\mathcal{C}_{\mathcal{W}} \subset \mathcal{W}$. The locations of the conjugates of the elements of each subset of $\mathcal{S}_0$ define how these subsets are interrelated and will guide the search for the sub-codebooks $\mathcal{C}_{\mathcal{X}}$, $\mathcal{C}_{\mathcal{Y} \cup \mathcal{Z}}$ and $\mathcal{C}_{\mathcal{W}}$ such that $|\mathcal{C}|$ is maximized.

We define the set of the i-th conjugates of the elements of the set $\mathcal{A}$ as $\mathcal{A}^{(i)}$ where:

$$\mathcal{A}^{(i)} \triangleq \{\mathbf{s}^{(i)} \ ; \ \mathbf{s} \in \mathcal{A}\}. \quad (33)$$

In this case, the relation $\mathcal{A}^{(i)} = \mathcal{B}$ indicates that the i-th conjugates of elements of $\mathcal{A}$ span the entire set $\mathcal{B}$ and, consequently, the i-th conjugation defines a one-to-one relationship between the sets $\mathcal{A}$ and $\mathcal{B}$. Evidently, $\mathcal{A}$ and $\mathcal{B}$ must have the same cardinality in this case. From the conjugates' properties in (26), the relation $\mathcal{A}^{(i)} = \mathcal{B}$ implies that $\mathcal{B}^{(\bar{i})} = \mathcal{A}$.

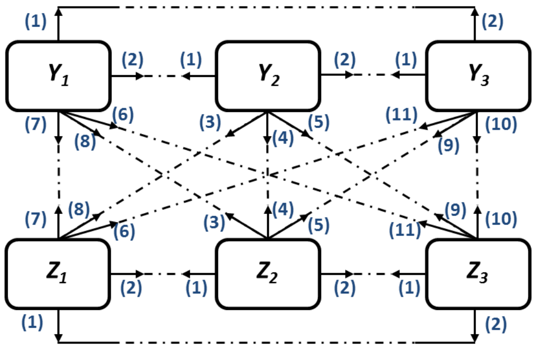

Fig. 4. Relations between the subsets of $\mathcal{Y}$ and $\mathcal{Z}$.

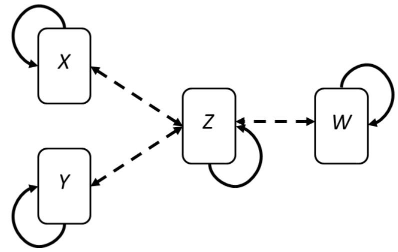

Fig. 5. Relations between the sets $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{W}$. Solid lines stand for the first two conjugates while dashed lines stand for the remaining conjugates.

1) The set $\mathcal{C}_{\mathcal{X}}$:

Observation *1:* The following bijections exist:

$$\mathcal{X}_1^{(1)} = \mathcal{X}_3 \ , \ \mathcal{X}_2^{(1)} = \mathcal{X}_1 \ , \ \mathcal{X}_3^{(1)} = \mathcal{X}_2. \quad (34)$$

Proof: From (24) and (29), it follows directly that $\mathcal{X}_1^{(1)} = \{\pi^{m+3}(1, \pi^{-3}(1), \pi^{-2}(1))\} = \mathcal{X}_3$, $\mathcal{X}_2^{(1)} = \{\pi^{m-1}(1, \pi^{1}(1), \pi^{2}(1))\} = \mathcal{X}_1$ and $\mathcal{X}_3^{(1)} = \{\pi^{m-1}(1, \pi^{1}(1), \pi^{-2}(1))\} = \mathcal{X}_2$ where $m \in \{0, \dots, M-1\}$ and the integers $m+3$ and $m-1$ (modulo-M) span this entire set. ■

Corollary *1:* Observation 1 implies that $|\mathcal{C}_{\mathcal{X}}| \leq M$.

Proof: Equation (34) results in $\mathcal{X}_1^{(2)} = \mathcal{X}_2$, $\mathcal{X}_2^{(2)} = \mathcal{X}_3$ and $\mathcal{X}_3^{(2)} = \mathcal{X}_1$ following from (26). Therefore, the inclusion of one element of $\mathcal{X}_i$ in $\mathcal{C}_{\mathcal{X}}$ implies that two elements (one from each of the remaining subsets $\mathcal{X}_j \ j \neq i$) need to be excluded from $\mathcal{C}_{\mathcal{X}}$ following from the one-to-one relationships between elements of $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_3$. Consequently, $2|\mathcal{C}_{\mathcal{X}}| \leq |\mathcal{X} \backslash \mathcal{C}_{\mathcal{X}}|$. Combining this inequality with $|\mathcal{C}_{\mathcal{X}}| + |\mathcal{X} \backslash \mathcal{C}_{\mathcal{X}}| = |\mathcal{X}| = 3M$ results in $|\mathcal{C}_{\mathcal{X}}| \leq M$. ■

2) The set $\mathcal{C}_{\mathcal{Y} \cup \mathcal{Z}}$:

Observation *2:* The following bijections hold:

$$\mathcal{Y}_i^{(1)} = \mathcal{Y}_j \ , \ \mathcal{Z}_i^{(1)} = \mathcal{Z}_j \ ; \ (i,j) = (1,3), (2,1), (3,2), \quad (35)$$

$$\begin{cases} \mathcal{Y}_1^{(6)} = \mathcal{Z}_3 \ , \ \mathcal{Y}_1^{(7)} = \mathcal{Z}_1 \ , \ \mathcal{Y}_1^{(8)} = \mathcal{Z}_2 \\ \mathcal{Y}_2^{(3)} = \mathcal{Z}_1 \ , \ \mathcal{Y}_2^{(4)} = \mathcal{Z}_2 \ , \ \mathcal{Y}_2^{(5)} = \mathcal{Z}_3 \\ \mathcal{Y}_2^{(9)} = \mathcal{Z}_2 \ , \ \mathcal{Y}_2^{(10)} = \mathcal{Z}_3 \ , \ \mathcal{Y}_2^{(11)} = \mathcal{Z}_1. \end{cases} \quad (36)$$

Proof: The proof follows from applying the conjugation in (24) and performing the suitable change of variable. We will provide the proof for two cases; the proof of the remaining cases follows in a similar manner. Let $m \in \{0, \dots, M-1\}$ in what follows. (i) $\mathcal{Y}_1^{(8)} = \mathcal{Z}_2$: From (24) and (30), $\mathcal{Y}_1^{(8)} = \{\pi^{m+n}(1, \pi^{1-n}(1), \pi^{-n-1}(1)) \ , \ n = 3, \dots, M-3\}$. Performing the change of variable $n' = 1 - n$ results in $n' = -2 \equiv M-2$ for $n = 3$ and $n' = 4 - M \equiv 4$ for $n = M-3$. Consequently, $\mathcal{Y}_1^{(8)} = \{\pi^{m+1-n'}(1, \pi^{n'}(1), \pi^{n'-2}(1) \ , \ n' = 4, \dots, M-2\} = \mathcal{Z}_2$ from (31) where $\{m+1-n' \ ; \ m = 0, \dots, M-1\} \equiv \{m \ ; \ m = 0, \dots, M-1\}$. (ii) $\mathcal{Y}_2^{(3)} = \mathcal{Z}_1$: From (24) and (30), $\mathcal{Y}_2^{(3)} = \{\pi^{m-1}(1, \pi^{n+1}(1), \pi^{1}(1)) \ , \ n = 2, \dots, M-4\}$. Performing the change of variable $n' = n+1$ results in $\mathcal{Y}_2^{(3)} = \{\pi^{m-1}(1, \pi^{n'}(1), \pi^{1}(1)) \ , \ n' = 3, \dots, M-3\} = \mathcal{Z}_1$ from (31) where $\{m-1 \ ; \ m = 0, \dots, M-1\} \equiv \{m \ ; \ m = 0, \dots, M-1\}$. ■

The bijections in (35)-(36), as well as the inverse bijections, are better highlighted in Fig. 4.

Corollary *2:* Observation 2 implies that $|\mathcal{C}_{\mathcal{Y} \cup \mathcal{Z}}| \leq M(M-5)$.

Proof: Equations (35)-(36) and Fig. 4 show that the conjugations define one-to-one relationships between any two distinct sets among the six sets $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3$. Therefore, the inclusion of one additional element of any of these six subsets in $\mathcal{C}_{\mathcal{Y} \cup \mathcal{Z}}$ incurs the exclusion of five other elements from the five remaining subsets following from the one-to-one relationships. Consequently, $5|\mathcal{C}_{\mathcal{Y} \cup \mathcal{Z}}| \leq |(\mathcal{Y} \cup \mathcal{Z}) \backslash \mathcal{C}_{\mathcal{Y} \cup \mathcal{Z}}|$. Combining this inequality with $|\mathcal{C}_{\mathcal{Y} \cup \mathcal{Z}}| + |(\mathcal{Y} \cup \mathcal{Z}) \backslash \mathcal{C}_{\mathcal{Y} \cup \mathcal{Z}}| = |\mathcal{Y} \cup \mathcal{Z}| = 6M(M-5)$ results in $|\mathcal{C}_{\mathcal{Y} \cup \mathcal{Z}}| \leq M(M-5)$. ■

Observation *3:* The following inclusion relations hold:

$$\mathcal{X}_j^{(i)} \subset \mathcal{Z} \ , \ \mathcal{Y}_j^{(i)} \subset \mathcal{Z} \ \ ; \ \ j = 1,2,3 \ , \ i \neq 1,2. \quad (37)$$

Proof: The proof is similar to the proofs of observations 1 and 2 and, hence, will be omitted. ■

3) The set $\mathcal{C}_{\mathcal{W}}$:

Observation *4:* The subsets of $\mathcal{W}$ satisfy the following relations:

$$\mathcal{W}_U^{(i)} = \mathcal{W}_U \ , \ \mathcal{W}_L^{(i)} = \mathcal{W}_L \ \ ; \ \ i = 1,2 \quad (38)$$

$$\begin{cases} \mathcal{W}_L^{(i)} \subset \mathcal{W}_U \\ \mathcal{W}_U^{(i)} \subset \mathcal{W}_L \cup \mathcal{Z} \end{cases} ; \ \ i = 3, \dots, 11. \quad (39)$$

Proof: The proof is provided in Appendix D. ■

The relations between the sets $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{W}$ are better highlighted in Fig. 5 following from observations 1, 2, 3 and 4.

4) The set $\mathcal{C}$:

Proposition *6:* The cardinality of $\mathcal{C}$ is maximized if $\mathcal{C}_{\mathcal{X}}$ is chosen to be equal to any one of the sets $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ and $\mathcal{C}_{\mathcal{Y} \cup \mathcal{Z}}$ is chosen to be equal to any one of the sets $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3$.

Proof: From corollary 1 and corollary 2:

$$|C| = |\mathcal{C}_{\mathcal{X}}| + |\mathcal{C}_{\mathcal{Y} \cup \mathcal{Z}}| + |\mathcal{C}_{\mathcal{W}}| \leq M + M(M-5) + |\mathcal{C}_{\mathcal{W}}|. \quad (40)$$

On the other hand, observation 4 shows that the conjugates of $\mathcal{W}$ are confined in $\mathcal{W} \cup \mathcal{Z}$. In a similar way, the conjugates of $\mathcal{X}$ are confined in $\mathcal{X} \cup \mathcal{Z}$ and that the conjugates of $\mathcal{Y}$ are confined in $\mathcal{Y} \cup \mathcal{Z}$ following from observations 1, 2 and 3. Therefore, the conjugates of $\mathcal{X}$ and $\mathcal{Y}$ can not belong to W and vice versa (refer to Fig. 5).

As a conclusion, the selection of $\mathcal{C}_{\mathcal{X}} = \mathcal{X}_i$ (for $i \in \{1,2,3\}$) and $\mathcal{C}_{\mathcal{Y} \cup \mathcal{Z}} = \mathcal{Y}_j$ (for $j \in \{1,2,3\}$) constitutes the best option for the interest of maximizing the cardinality. (i): This selection achieves the maximum limit of $\max\{|\mathcal{C}_\mathcal{X}|+|\mathcal{C}_{\mathcal{Y}\cup\mathcal{Z}}|\} = M+M(M-5)$ since $|\mathcal{X}_i|=M$ and $|\mathcal{Y}_j|=M(M-5)$ for all values of i and j. (ii): From observation 1 (resp. 2), the first two conjugates of $\mathcal{X}_i$ (resp. $\mathcal{Y}_j$) are in $\mathcal{X}_{i'}$ for $i'\neq i$ (resp. $\mathcal{Y}_{j'}$ for $j'\neq j$) while the remaining conjugates are always in $\mathcal{Z}$ (from observation 3) implying that the conjugates of the set $\mathcal{X}_i\cup\mathcal{Y}_j$ are always outside this set and outside $\mathcal{C}_\mathcal{W}\subset\mathcal{W}$. (iii): At the same time, this selection does not constrain the selection of $\mathcal{C}_\mathcal{W}$ in $\mathcal{W}$ since the conjugates of $\mathcal{X}_i\cup\mathcal{Y}_j$ can not belong to $\mathcal{W}$. In this context, while the inclusion of elements of $\mathcal{Z}$ in $\mathcal{C}_\mathcal{X}\cup\mathcal{C}_{\mathcal{Y}\cup\mathcal{Z}}$ might allow to achieve the maximum value of $\max\{|\mathcal{C}_\mathcal{X}|+|\mathcal{C}_{\mathcal{Y}\cup\mathcal{Z}}|\}$, this approach presents the limitation of constraining the search for $\mathcal{C}_\mathcal{W}$ (since conjugates of $\mathcal{Z}$ might belong to $\mathcal{W}$) resulting eventually in a smaller cardinality. ■

From proposition 6, the codebooks with maximum cardinality can be constructed as follows:

$$\mathcal{C}=\mathcal{X}_i\cup\mathcal{Y}_j\cup\mathcal{C}_\mathcal{W}\ ;\ i,j\in\{1,2,3\}, \tag{41}$$

where the search for $\mathcal{C}_\mathcal{W}$ has been decoupled from the search for $\mathcal{C}_\mathcal{X}\cup\mathcal{C}_{\mathcal{Y}\cup\mathcal{Z}}$. Solving for $\mathcal{C}_\mathcal{W}$ is constrained by the conjugates that belong to $\mathcal{W}$ since, from (38)-(39), these conjugates belong to $\mathcal{W}\cup\mathcal{Z}$ while $\mathcal{Z}$ is excluded from the construction.

D. Solving for $\mathcal{C}_\mathcal{W}$

We next solve for the set $\mathcal{C}_\mathcal{W}$ with maximum cardinality. Following from (38), $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$ belong to $\mathcal{W}_U$ (resp. $\mathcal{W}_L$) when $\mathbf{s}$ belongs to $\mathcal{W}_U$ (resp. $\mathcal{W}_L$). In what follows, k is defined as the unique integer such that M is written as $M=3k$, $M=3k+1$ or $M=3k-1$.

We next check if any of the first two conjugates can be equal to $\mathbf{s}$. Note that $\mathbf{s}^{(1)}=\mathbf{s}$ implies that $\mathbf{s}^{(2)}=\mathbf{s}$ and vice versa following from (25).

Observation 5: The relation $\mathbf{s}=\mathbf{s}^{(1)}=\mathbf{s}^{(2)}$ can hold if and only if ($m=0,\dots,M-1$):

$$\mathbf{s}\triangleq\mathbf{s}_{\text{self}}=\begin{cases}\pi^m\left(1,\pi^k(1),\pi^{2k}(1)\right)\in\mathcal{W}_U, & \text{if}\quad M=3k+1;\\ \pi^m\left(1,\pi^{2k-1}(1),\pi^{k-1}(1)\right)\in\mathcal{W}_L, & \text{if}\quad M=3k-1.\end{cases} \tag{42}$$

Proof: The proof is provided in Appendix E. ■

Evidently, symbols satisfying (42) must be excluded from $\mathcal{C}_\mathcal{W}$. If M is a multiple of 3, then none of the elements can be equal to the first or second conjugate of itself. As a conclusion, the construction needs to be limited to the sets $\mathcal{W}'_U\triangleq\mathcal{W}_U\backslash\{\mathbf{s}_{\text{self}}\}$ and $\mathcal{W}'_L\triangleq\mathcal{W}_L\backslash\{\mathbf{s}_{\text{self}}\}$. From (32), if follows that $|\mathcal{W}_U|=M\frac{(M-5)(M-6)}{2}$ and $|\mathcal{W}_L|=M\frac{(M-6)(M-7)}{2}$ resulting in:

$$|\mathcal{W}'_U|=M\begin{cases}\frac{(M-5)(M-6)}{2}-1, & M=3k+1;\\ \frac{(M-5)(M-6)}{2}, & M=3k\text{ or }M=3k-1.\end{cases} \tag{43}$$

and:

$$|\mathcal{W}'_L|=M\begin{cases}\frac{(M-6)(M-7)}{2}-1, & M=3k-1;\\ \frac{(M-6)(M-7)}{2}, & M=3k\text{ or }M=3k+1.\end{cases} \tag{44}$$

where it can be easily proven that $|\mathcal{W}'_U|$ and $|\mathcal{W}'_L|$ are always divisible by 3 for all values of M.

Given that the first and second conjugates of the elements of $\mathcal{W}'_U$ are always confined in $\mathcal{W}'_U$, then this set can be partitioned into three subsets (of the same cardinality) such that one subset contains $\frac{|\mathcal{W}'_U|}{3}$ elements while the other two subsets contain the first and second conjugates of these elements, respectively. The same holds for $\mathcal{W}'_L$ that can partitioned into three subsets of cardinality $\frac{|\mathcal{W}'_L|}{3}$ each such that if an element belongs to a certain subset, then its first and second conjugates fall in the remaining two subsets.

Observation 6: If $\mathcal{W}'_U$ is partitioned as $\mathcal{W}'_U=\mathcal{W}'_{U,h}\cup\mathcal{W}'_{U,v}\cup\mathcal{W}'_{U,d}\triangleq\left[\cup_{n_1=2}^{k-1}W'_{U,h,n_1}\right]\cup\left[\cup_{n_1=2}^{k-1}W'_{U,v,n_1}\right]\cup\left[\cup_{n_1=2}^{k-1}W'_{U,d,n_1}\right]$ where ($m=0,\dots,M-1$):

$$W'_{U,h,n_1}=\{\pi^m(1,\pi^{n_1}(1),\pi^{n_2}(1))\ ;\ n_2=2n_1,\dots,(M-2)-n_1\}, \tag{45}$$

$$W'_{U,v,n_1}=\{\pi^m(1,\pi^{n_2}(1),\pi^{-n_1-1}(1))\ ;\ n_2=n_1,\dots,(M-2)-2n_1\}, \tag{46}$$

$$W'_{U,d,n_1}=\{\pi^m(1,\pi^{n_2}(1),\pi^{n_1+n_2}(1))\ ;\ n_2=n_1+1,\dots,(M-1)-2n_1\}, \tag{47}$$

then $\mathcal{W}'^{(1)}_{U,h}=\mathcal{W}'_{U,d}$ and $\mathcal{W}'^{(2)}_{U,h}=\mathcal{W}'_{U,v}$ implying that, for any element $\mathbf{s}$ of $\mathcal{W}'_U$, the elements $\mathbf{s}$, $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$ belong to three different subsets among $\mathcal{W}'_{U,h}$, $\mathcal{W}'_{U,v}$ and $\mathcal{W}'_{U,d}$.

Proof: The proof is provided in Appendix F. ■

Note that, from (25)-(26), the relation $(\mathcal{W}'^{(1)}_{U,h},\mathcal{W}'^{(2)}_{U,h})=(\mathcal{W}'_{U,d},\mathcal{W}'_{U,v})$, implies that $(\mathcal{W}'^{(1)}_{U,v},\mathcal{W}'^{(2)}_{U,v})=(\mathcal{W}'_{U,h},\mathcal{W}'_{U,d})$ and $(\mathcal{W}'^{(1)}_{U,d},\mathcal{W}'^{(2)}_{U,d})=(\mathcal{W}'_{U,v},\mathcal{W}'_{U,h})$. Note also that:

$$|\mathcal{W}'_{U,h}|=|\mathcal{W}'_{U,v}|=|\mathcal{W}'_{U,d}|=\frac{|\mathcal{W}'_U|}{3}=M\left\lfloor\frac{(M-5)(M-6)}{6}\right\rfloor, \tag{48}$$

where $\lfloor x\rfloor$ rounds x to the nearest integer that is smaller than x.

Observation 7: If $\mathcal{W}'_L$ is partitioned as $\mathcal{W}'_L=\mathcal{W}'_{L,h}\cup\mathcal{W}'_{K,v}\cup\mathcal{W}'_{K,d}\triangleq\left[\cup_{n_1=2}^{k'-1}W'_{L,h,n_1}\right]\cup\left[\cup_{n_1=2}^{k'-1}W'_{L,v,n_1}\right]\cup\left[\cup_{n_1=2}^{k'-1}W'_{L,d,n_1}\right]$ where $k'=k-1$ if $M=3k-1$ and $k'=k$ if $M=3k$ or $M=3k+1$ and ($m=0,\dots,M-1$):

$$W'_{L,h,n_1}=\{\pi^m(1,\pi^{-n_1-1}(1),\pi^{n_2}(1))\ ;\ n_2=n_1,\dots,(M-3)-2n_1\}, \tag{49}$$

$$W'_{L,v,n_1}=\{\pi^m(1,\pi^{n_2}(1),\pi^{n_1}(1))\ ;\ n_2=2n_1+1,\dots,(M-2)-n_1\}, \tag{50}$$

$$W'_{L,d,n_1}=\{\pi^m(1,\pi^{n_1+n_2+1}(1),\pi^{n_2}(1))\ ;\ n_2=n_1+1,\dots,(M-2)-2n_1\}, \tag{51}$$

then $\mathcal{W}'^{(1)}_{L,h}=\mathcal{W}'_{L,d}$ and $\mathcal{W}'^{(2)}_{L,h}=\mathcal{W}'_{L,v}$ implying that, for any element $\mathbf{s}$ of $\mathcal{W}'_L$, the elements $\mathbf{s}$, $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$ belong to three different subsets among $\mathcal{W}'_{L,h}$, $\mathcal{W}'_{L,v}$ and $\mathcal{W}'_{L,d}$.

Proof: The proof is similar to the proof of observation 6 and, hence, will be omitted. ■

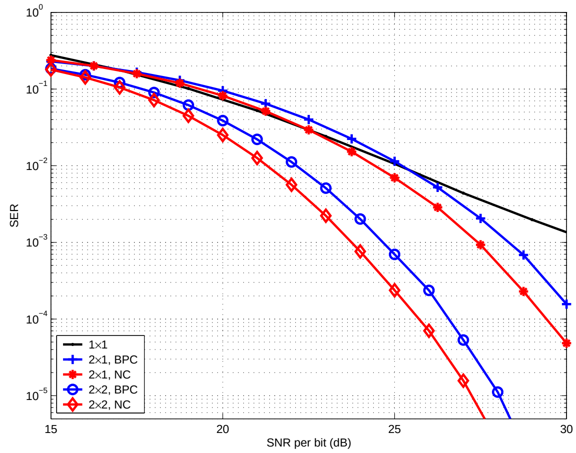

Fig. 6. Performance of the new-code (NC) versus the best-previous-code (BPC) with 2 transmit antennas and 4-PPM.

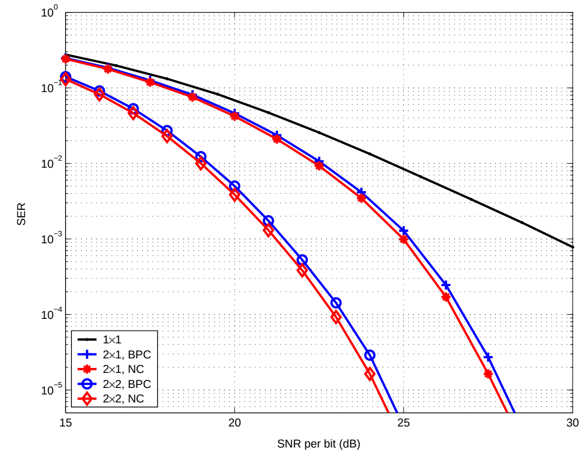

Fig. 7. Performance of the new-code (NC) versus the best-previous-code (BPC) with 2 transmit antennas and 8-PPM.

Proposition 7: The maximum possible cardinality of $\mathcal{C}_{\mathcal{W}}$ is:

$$\max\{|\mathcal{C}_{\mathcal{W}}|\} = \frac{|\mathcal{W}'_U|}{3} = M\left\lfloor\frac{(M-5)(M-6)}{6}\right\rfloor. \tag{52}$$

Proof: The proof is provided in Appendix G. ■

Corollary 3: From proposition 7, the set $\mathcal{C}_{\mathcal{W}}$ with maximum cardinality can be selected as:

$$\mathcal{C}_{\mathcal{W}} = \mathcal{W}'_{U,h} \text{ or } \mathcal{C}_{\mathcal{W}} = \mathcal{W}'_{U,v} \text{ or } \mathcal{C}_{\mathcal{W}} = \mathcal{W}'_{U,d}. \tag{53}$$

Proof: If $\mathcal{C}_{\mathcal{W}}$ is taken to be equal to any subset among the three subsets $\mathcal{W}'_{U,h}$, $\mathcal{W}'_{U,v}$ or $\mathcal{W}'_{U,d}$, then the first and second conjugates will fall in the two other subsets following from observation 6 while the remaining conjugates will be outside $\mathcal{W}_U$ following from (39). Therefore, the conjugates of elements of $\mathcal{C}_{\mathcal{W}}$ will be outside $\mathcal{C}_{\mathcal{W}}$ while, from (48), the cardinality will be equal to the maximum cardinality in (52). ■

Note that a number of solutions other than (53) can be obtained where an element of a subset among $\mathcal{W}'_{U,h}$, $\mathcal{W}'_{U,v}$ and $\mathcal{W}'_{U,d}$ can be replaced by either its first conjugate or its second conjugate from the other two subsets. Finally, from (41) and (53), selecting $i = 1$, $j = 1$ and $\mathcal{C}_{\mathcal{W}} = \mathcal{W}'_{U,h}$ results in $\mathcal{C} = \mathcal{X}_1 \cup \mathcal{Y}_1 \cup \mathcal{W}'_{U,h}$. Replacing $\mathcal{X}_1$, $\mathcal{Y}_1$ and $\mathcal{W}'_{U,h} = \cup_{n_1=2}^{k-1} W'_{U,h,n_1}$ by their values from (29), (30) and (45), respectively, results in the solution provided in (10).

VIII. Numerical Results

Simulations are performed over the IEEE 802.15.3a UWB channel model CM2 [23]. The integration time and filter bandwidth are set to $T_i = 5$ ns and $W = 5$ GHz, respectively. The new-codes (NC) are compared with the best-previous-codes (BPC) in [20] where these codes outperform other schemes that satisfy comparable construction constraints (for example, the reader is referred to figure 6 in [20]). Moreover, the comparison with [20] is judged to be the most illustrative since the compared codes satisfy exactly the same set of constraints. Results show the variation of the Symbol-Error-Rate (SER) as a function of the SNR per information bit. In what follows, the curves labeled 1×1 correspond to the SER of SISO non-coherent systems with M-PPM. For such systems, after integrating the energy in the M slots, the ED receiver decides in favor of the PPM slot containing the maximum energy.

Fig. 6 and Fig. 7 show the performance with two transmit antennas for $M = 4$ and $M = 8$, respectively. Results highlight the high performance levels that can be achieved by the proposed code. Regarding the diversity order, the diversity gains with respect to single-antenna systems are evident while the proposed code and [20] achieve the same diversity order where the corresponding SER curves are practically parallel to each other for large SNRs. Results show the superiority of the proposed code with respect to [20] over the entire SNR range and for different numbers of receive antennas. The performance gains are in the order of 1 dB and 0.25 dB for $M = 4$ and $M = 8$, respectively. It is worth noting that the performance of M-PPM enhances with M despite the increase in the data rate [3]. This clearly distinguishes this M-dimensional modulation compared to the more conventional two-dimensional QAM and PSK modulations.

Fig. 8 and Fig. 9 show the performance with three transmit antennas for $M = 5$ and $M = 8$, respectively. In this case, the performance gains with respect to [20] are in the order of 0.9 dB and 0.3 dB for $M = 5$ and $M = 8$, respectively.

It is worth noting that the performance gains in figures 6, 7, 8 and 9 can be achieved while respecting the same construction constraints as in [20]. In other words, these can be perceived as complimentary gains that are not associated with any consequential limitations.

IX. Conclusion

We have proposed novel encoding schemes that can be associated with 2×2 and 3×3 noncoherent MIMO IR-UWB systems. The proposed schemes are characterized by a remarked simplicity where each transmit antenna is pulsed once per symbol while the receiver is based on analog energy detection in a way that is completely analogous to noncoherent SISO IR-UWB systems. The main advantage of the proposed constructions resides in attaining the maximum achievable

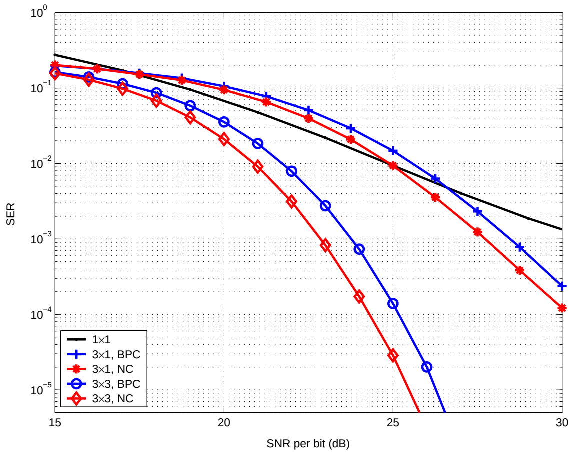

Fig. 8. Performance of the new-code (NC) versus the best-previous-code (BPC) with 3 transmit antennas and 5-PPM.

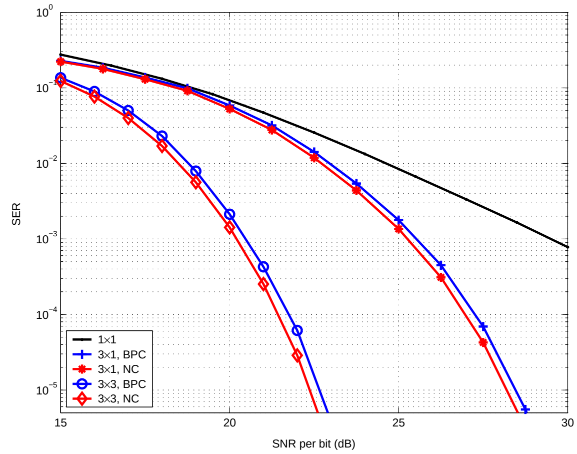

Fig. 9. Performance of the new-code (NC) versus the best-previous-code (BPC) with 3 transmit antennas and 8-PPM.

code rates under the PPM-shaping and ED-decodability constraints. Future work must consider possible extensions to higher numbers of transmit antennas. The expected increase in the number of conjugates renders solving for the maximum-rate codebooks much more challenging.

Appendix A

Consider the relation $(I_{1,3}, I_{2,3}, I_{3,3}) = (1,1,1) \Leftrightarrow (l_8 \vee l_{13} \vee l_{14}) \wedge (l_2 \vee l_5 \vee l_{15}) \wedge (l_1 \vee l_6 \vee l_{12})$ where this logical statement can be expanded as $\sum_{i\in\{8,13,14\}} \sum_{j\in\{2,5,15\}} \sum_{k\in\{1,6,12\}} l_i \wedge l_j \wedge l_k$ where this triple summation contains 27 terms. Among these terms, 22 terms contradict (16) or result in $\mathbf{s}' = \mathbf{s}$ while the remaining terms will result in five of the conjugates in (18). The first category of terms includes, for example, $l_{13} \wedge l_{15} \wedge l_6 \Leftrightarrow s_1 = \pi(s_3)$, $l_{14} \wedge l_2 \wedge l_{12} \Leftrightarrow s_2 = s_3$, $l_8 \wedge l_5 \wedge l_1 \Leftrightarrow \mathbf{s}' = \mathbf{s}$ and $l_{13} \wedge l_{15} \wedge l_{12} \Leftrightarrow s_1 = s_2$ all contradicting (16).

The five terms that do not contradict criterion 1 or the condition $\mathbf{s}' = \mathbf{s}$ are $(l_{13} \wedge l_{15} \wedge l_1)$, $(l_8 \wedge l_2 \wedge l_6)$, $(l_{14} \wedge l_5 \wedge l_{12})$, $(l_{14} \wedge l_{15} \wedge l_6)$ and $(l_{13} \wedge l_2 \wedge l_{12})$. Consider the first statement $l_{13} \wedge l_{15} \wedge l_1$ that results in $\mathbf{s}' = (s'_1, s'_2, s'_3) = (s_1, s_3, \pi^{-1}(s_2)) = \mathbf{s}^{(5)}$ given in (18). For $\mathbf{s}' = \mathbf{s}^{(5)}$ to be an element of $\mathcal{C}$, it must satisfy the conditions in (16). (i): The condition $s'_1 \neq s'_2$ holds since it implies that $s_1 \neq s_3$ where this condition holds since $\mathbf{s}$ satisfies (16). (ii): $s'_2 \neq s'_3$ holds since $s_3 \neq \pi^{-1}(s_2) \Leftrightarrow s_2 \neq \pi(s_3)$ from (16). (iii): $s'_1 \neq \pi(s'_3)$ holds since $s_1 \neq \pi(\pi^{-1}(s_2)) = s_2$ from (16). (iv): $s'_2 \neq \pi(s'_3)$ holds since $s_3 \neq \pi(\pi^{-1}(s_2)) = s_2$ from (16). (v): $s'_1 \neq \pi(s'_2)$ holds since $s_1 \neq \pi(s_3)$ from (16). While the five previous relations always hold for any $\mathbf{s} \in \mathcal{C}$, the relation $s'_1 \neq s'_3$ does not always hold. In fact, $s'_1 \neq s'_3 \Leftrightarrow s_1 \neq \pi^{-1}(s_2) \Leftrightarrow s_2 \neq \pi(s_1)$ where the last inequality does not appear in (16). Consequently, for some elements of $\mathcal{C}$, s_2 might be equal to $\pi(s_1)$. Therefore, for $\mathbf{s}^{(5)}$ to belong to $\mathcal{C}$, the condition $s_2 \neq \pi(s_1)$ must be satisfied. The analysis of the four remaining statements follows in a similar manner.

Appendix B

The set $\mathcal{S}^2$ can be written as $\mathcal{S}^2 = \{(x, \pi^n(x)); x = 1 \cdots M, n = 0 \cdots M-1\}$. From (11), elements of $\mathcal{S}_0$ must satisfy $x \neq \pi^n(x)$ and $x \neq \pi^{n+1}(x)$ implying that n and $n+1$ must be different from $[0 \mod M]$. Consequently, $\mathcal{S}_0$ can be written as $\mathcal{S}_0 = \mathcal{S}_0^{(1)} \cup \mathcal{S}_0^{(2)} \triangleq \{(x, \pi^{n-1}(x))\ ,\ x = 1 \cdots M\}_{n=2}^{M/2} \cup \{(x, \pi^{n-1}(x))\ ,\ x = 1 \cdots M\}_{n=M/2+1}^{M-1}$. Setting $x = \pi^m(1)$ for $m = 0, \ldots, M-1$ in $\mathcal{S}_0^{(1)}$ implies that elements of this set can be written as $(\pi^m(1), \pi^{m+n-1}(1)) = (\pi^m(1), \pi^m(n)) = \pi^m(1, n)$ since $\pi^{n-1}(1) = n$ for $n = 2, \ldots, \frac{M}{2}$. Performing the change of variable $n \rightarrow M - n + 1$ in $\mathcal{S}_0^{(2)}$ implies that this set can be written as $\{(x, \pi^{M-n}(x))\ ,\ x = 1 \cdots M\}_{n=2}^{M/2}$. Setting $x = \pi^{m+n-1}(1)$ for $m = 0, \ldots, M-1$ implies that elements of this set can be written as $(\pi^{m+n-1}(1), \pi^{M+m-1}(1)) = (\pi^m(n), \pi^{m-1}(1)) = \pi^m(n, M)$ since $\pi^{-1}(1) = M$ and $\pi^M(s) = s$.

Appendix C

Observation 8: The following relation holds:

$$\forall\ \mathbf{s} \in \mathcal{S}_0, i \in \{3, \ldots, 11\}\ \ \exists\ \ \bar{i} \in \{3, \ldots, 11\}\ |\ \left[\mathbf{s}^{(i)}\right]^{(\bar{i})} = \mathbf{s}. \tag{54}$$

In particular, $\bar{i} = 8, 4, 9, 11, 7, 3, 5, 10, 6$ for $i = 3, 4, 5, 6, 7, 8, 9, 10, 11$, respectively.

Proof: We provide the proof for $i = 3$ and the remaining cases follow in a similar manner. Let $\mathbf{s}' = (s'_1, s'_2, s'_3) = \mathbf{s}^{(3)} = (\pi(s_3), s_2, s_1)$. From (18), $\mathbf{s}'^{(8)} = (s'_3, s'_2, \pi^{-1}(s'_1)) = (s_1, s_2, s_3) = \mathbf{s}$ where $\mathbf{s}'^{(8)}$ exists since $s'_3 \neq \pi(s'_2)$ following from $s_1 \neq \pi(s_2)$ which follows from (16). ■

Proposition 8: Let $\mathcal{C}$ be a codebook with maximum cardinality in $\mathcal{S}_0$. The set $\mathcal{S}_0$ can be partitioned as $\mathcal{S}_0 = \mathcal{C} \cup \mathcal{C}'$ where $\mathcal{C}' \triangleq \{\mathbf{s}^{(1)}, \ldots, \mathbf{s}^{(11)}\ |\ \mathbf{s} \in \mathcal{C}\}$ contains the conjugates of all the codewords. In other words, there are no elements of $\mathcal{S}_0$ that are neither codewords (of the codebook with maximum cardinality) nor conjugates of codewords.

Proof: Assume that $\mathcal{S}_0 \backslash (\mathcal{C} \cup \mathcal{C}')$ is not empty and denote by $\mathbf{s}$ an element of this set. We will next prove that $\mathbf{s}$ can be always moved into $\mathcal{C}$.

Consider first the conjugates $\mathbf{s}^{(3)},\ldots,\mathbf{s}^{(11)}$. Assume that $\exists\ i \in \{3,\ldots,11\}\ |\ \mathbf{s}^{(i)} \in \mathcal{C}$. From observation 8, $\exists\ \bar{i}\ |\ \left[\mathbf{s}^{(i)}\right]^{(\bar{i})} = \mathbf{s}$ implying that $\mathbf{s}$ is the $\bar{i}$-th conjugate of an element of $\mathcal{C}$ ($\mathbf{s}^{(i)} \in \mathcal{C}$) and, consequently, must belong to $\mathcal{C}'$ contradicting the fact that $\mathbf{s} \in \mathcal{S}_0 \backslash (\mathcal{C} \cup \mathcal{C}')$. Therefore:

$$\mathbf{s}^{(3)},\ldots,\mathbf{s}^{(11)} \notin \mathcal{C}. \tag{55}$$

Now, consider the remaining conjugates $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$. The following possibilities arise.

Case 1: $\mathbf{s}^{(1)} \in \mathcal{C}$ and $\mathbf{s}^{(2)} \in \mathcal{C}$. This case is impossible since $\left[\mathbf{s}^{(1)}\right]^{(1)} = \mathbf{s}^{(2)}$ must be outside $\mathcal{C}$ for $\mathbf{s}^{(1)}$ to be in $\mathcal{C}$.

Case 2: $\mathbf{s}^{(1)} \notin \mathcal{C}$ and $\mathbf{s}^{(2)} \notin \mathcal{C}$. Combining these relations with (55) results in $\mathbf{s}^{(1)},\ldots,\mathbf{s}^{(11)} \notin \mathcal{C}$ and, hence, $\mathbf{s}$ can be included in $\mathcal{C}$ (since its conjugates are all outside $\mathcal{C}$) thus contradicting the fact that $\mathcal{C}$ has the maximum cardinality.

Case 3: $\mathbf{s}^{(i)} \notin \mathcal{C}$ and $\mathbf{s}^{(i')} \in \mathcal{C}$ for $(i,i') = (1,2)$ or $(2,1)$. In this case, by moving $\mathbf{s}^{(i')}$ to $\mathcal{C}'$ and $\mathbf{s}$ to $\mathcal{C}$ (since $\mathbf{s}^{(1)},\ldots,\mathbf{s}^{(11)}$ will be outside $\mathcal{C}$), the same maximum cardinality is maintained while increasing the number of elements of $\mathcal{C}'$. This alternative solution of maintaining the same maximum cardinality of $\mathcal{C}$ while increasing the cardinality of $\mathcal{C}'$ is more desirable since $\mathbf{s}^{(i')}$ can concurrently serve as the conjugate to codewords other than $\mathbf{s}$ in $\mathcal{C}$.

As a conclusion, all elements of $\mathcal{S}_0 \backslash (\mathcal{C} \cup \mathcal{C}')$ can be moved to $\mathcal{C}$ (since their conjugates will be outside $\mathcal{C}$) for the sake of maximizing the size of the codebook while respecting the structure in (19). Therefore, $\mathcal{S}_0 \backslash (\mathcal{C} \cup \mathcal{C}')$ is empty and $\mathcal{S}_0 = \mathcal{C} \cup \mathcal{C}'$. ■

Corollary *4:* If $\mathbf{s} \in \mathcal{C}$ and $\mathbf{s}' \in \mathcal{C}$, then $\pi^m(\mathbf{s})$ and $\pi^m(\mathbf{s}')$ are elements of $\mathcal{C}$.

Proof: From proposition 8, $\mathbf{s} \in \mathcal{C} \Rightarrow \mathbf{s}^{(1)},\ldots,\mathbf{s}^{(11)} \in \mathcal{C}'$ and $\mathbf{s}' \in \mathcal{C} \Rightarrow \mathbf{s}'^{(1)},\ldots,\mathbf{s}'^{(11)} \in \mathcal{C}'$. Now, $\mathbf{s} \neq \mathbf{s}'^{(i)}$ for $i = 1,\ldots,11$ since $\mathbf{s} \in \mathcal{C}$ and $\mathbf{s}'^{(i)} \in \mathcal{C}'$. Taking the permutation of order m of both sides of the inequality implies that $\pi^m(\mathbf{s}) \neq \pi^m(\mathbf{s}'^{(i)}) = [\pi^m(\mathbf{s}')]^{(i)}$. In a similar manner, $\mathbf{s}' \neq \mathbf{s}^{(i)}$ implying that $\pi^m(\mathbf{s}') \neq \pi^m(\mathbf{s}^{(i)}) = [\pi^m(\mathbf{s})]^{(i)}$. From proposition 8, the relations $\pi^m(\mathbf{s}) \neq [\pi^m(\mathbf{s}')]^{(i)}$ and $\pi^m(\mathbf{s}') \neq [\pi^m(\mathbf{s})]^{(i)}$ imply that the elements $\pi^m(\mathbf{s})$ and $\pi^m(\mathbf{s}')$ can be readily selected as codewords (and their conjugates will be readily in $\mathcal{C}'$). Therefore, adding $\pi^m(\mathbf{s})$ and $\pi^m(\mathbf{s}')$ to $\mathcal{C}$ can be accomplished without violating any condition. As a conclusion, these elements must be added since $\mathcal{C}$ is the codebook having maximum cardinality. ■

Appendix D

From (32), elements of $\mathcal{W}$ can be written as $\mathbf{s} = \pi^m(1, \pi^{n_1}(1), \pi^{n_2}(1))$ where the following conditions are satisfied for $\mathcal{W}_U$:

$$U_1 : n_1 \in \{2,\ldots,M-5\} \quad , \quad U_2 : n_2 \in \{4,\ldots,M-3\} \quad ,$$
$$U_3 : n_2 - n_1 \in \{2,\ldots,M-5\}, \tag{56}$$

and the following conditions are satisfied for $\mathcal{W}_L$:

$$L_1 : n_1 \in \{5,\ldots,M-3\} \quad , \quad L_2 : n_2 \in \{2,\ldots,M-6\} \quad ,$$
$$L_3 : n_1 - n_2 \in \{3,\ldots,M-5\}. \tag{57}$$

We will next analyze the set $\mathcal{W}_U^{(i)}$ for $i = 1,\ldots,11$. The analysis of $\mathcal{W}_L^{(i)}$ follows in a similar manner. Consider an element $\mathbf{s}$ of $\mathcal{W}_U$, the conjugate $\mathbf{s}^{(i)}$ can be written as $\pi^m(1, \pi^{N_1}(1), \pi^{N_2}(1))$ and the following cases need to be considered.

$i = 1$: $N_1 = -n_2 - 1$ and $N_2 = n_1 - n_2 - 1$ from (24). From U_2, $-n_2 \in \{3,\ldots,M-4\} \Rightarrow N_1 \in \{2,\ldots,M-5\}$ implying that N_1 satisfies U_1. From U_3, $n_1 - n_2 \in \{5,\ldots,M-2\} \Rightarrow N_2 \in \{4,\ldots,M-3\}$ implying that N_2 satisfies U_2. From U_1, $N_2 - N_1 = n_1 \in \{2,\ldots,M-5\}$ implying that $N_2 - N_1$ satisfies U_3. Therefore, $\mathbf{s}^{(1)} \in \mathcal{W}_U$.

$i = 2$: $N_1 = n_2 - n_1$ and $N_2 = -n_1 - 1$ from (24). From U_3, $N_1 \in \{2,\ldots,M-5\}$ implying that N_1 satisfies U_1. From U_1, $N_2 \in \{4,\ldots,M-3\}$ implying that N_2 satisfies U_2. From U_2, $N_2 - N_1 = -n_2 - 1 \in \{2,\ldots,M-5\}$ implying that $N_2 - N_1$ satisfies U_3. Therefore, $\mathbf{s}^{(2)} \in \mathcal{W}_U$.

$i = 3$: $N_1 = n_1 - n_2 - 1$ and $N_2 = -n_2 - 1$ from (24). Case 1: if $n_1 = 2$, then $N_1 = 1 - n_2$ and $N_2 = -n_2 - 1$ where $N_1 \in \{4,\ldots,M-3\}$ following from U_2. In this case, $N_1 - N_2 = 2$ and $\mathbf{s}^{(3)} \in \mathcal{Z}_2$ given in (31). Case 2: $n_1 \neq 2$. Combining this condition with (56) results in the following new set of conditions:

$$U_1' : n_1 \in \{3,\ldots,M-5\} \quad , \quad U_2' : n_2 \in \{5,\ldots,M-3\} \quad ,$$
$$U_3' : n_2 - n_1 \in \{2,\ldots,M-6\}, \tag{58}$$

From U_3', $N_1 \in \{5,\ldots,M-3\}$ implying that N_1 satisfies L_1. From U_2', $-n_2 \in \{3,\ldots,M-5\} \Rightarrow N_2 \in \{2,\ldots,M-6\}$ implying that N_2 satisfies L_2. From U_1', $N_1 - N_2 = n_1 \in \{3,\ldots,M-5\}$ implying that $N_1 - N_2$ satisfies L_3. As a conclusion, $\mathbf{s}^{(3)} \in \mathcal{W}_L$ for $n_1 \neq 2$.

In a way that is analogous to the case $i = 3$, it can be proven that the conjugates of $\mathbf{s} \in \mathcal{W}_U$ satisfy the following relations. (i): $(\mathbf{s}^{(3)}, \mathbf{s}^{(4)}, \mathbf{s}^{(5)}) \in \mathcal{Z}_2 \times \mathcal{Z}_3 \times \mathcal{Z}_1$ for $n_1 = 2$ and $(\mathbf{s}^{(3)}, \mathbf{s}^{(4)}, \mathbf{s}^{(5)}) \in \mathcal{W}_L^3$ otherwise. (ii): $(\mathbf{s}^{(6)}, \mathbf{s}^{(7)}, \mathbf{s}^{(8)}) \in \mathcal{Z}_1 \times \mathcal{Z}_2 \times \mathcal{Z}_3$ for $n_2 - n_1 = 2$ and $(\mathbf{s}^{(6)}, \mathbf{s}^{(7)}, \mathbf{s}^{(8)}) \in \mathcal{W}_L^3$ otherwise. (iii): $(\mathbf{s}^{(9)}, \mathbf{s}^{(10)}, \mathbf{s}^{(11)}) \in \mathcal{Z}_3 \times \mathcal{Z}_1 \times \mathcal{Z}_2$ for $n_2 = M - 3$ and $(\mathbf{s}^{(9)}, \mathbf{s}^{(10)}, \mathbf{s}^{(11)}) \in \mathcal{W}_L^3$ otherwise. As a conclusion, $\mathcal{W}_U^{(i)} = \mathcal{W}_U$ for $i = 1, 2$ and $\mathcal{W}_U^{(i)} \subset (\mathcal{W}_L \cup \mathcal{Z})$ for $i = 3,\ldots,11$.

Appendix E

We will solve $\mathbf{s}^{(1)} = \pi^{m+n_2+1}(1, \pi^{-n_2-1}(1), \pi^{n_1-n_2-1}(1)) = \pi^{m'}(1, \pi^{n_1}(1), \pi^{n_2}(1)) = \mathbf{s}$. This results in the following equations: $(m - m') + n_2 + 1 \equiv 0$, $(m - m') \equiv n_1$ and $(m - m') + n_1 \equiv n_2$. Eliminating $(m - m')$ results in the two following equations: $2n_1 \equiv n_2$ and $n_1 + n_2 \equiv -1$.

Adding the last two equations results in $3n_1 \equiv -1$. (i): If $M = 3k$, $3n_1 \equiv -1 \Rightarrow 3n_1 = k'(3k) - 1 \Rightarrow n_1 = k'k - \frac{1}{3}$ which is not an integer. (ii): If $M = 3k + 1$, then it is possible to have $3n_1 \equiv -1$ if $3n_1 = (3k + 1) - 1 = 3k$ resulting in $n_1 = k$ and $n_2 = 2k$ where the obtained element will belong to $\mathcal{W}_U$ from (32). (iii): If $M = 3k - 1$, then it is possible to have $3n_1 \equiv -1$ if $3n_1 = 2(3k - 1) - 1 = 6k - 3$ resulting in $n_1 = 2k - 1$ and $n_2 = 4k - 2 \equiv k - 1$ where the obtained element will belong to $\mathcal{W}_L$ from (32).

Appendix F

First, we check that $|\mathcal{W}'_{U,h}| + |\mathcal{W}'_{U,v}| + |\mathcal{W}'_{U,d}| = |\mathcal{W}'_U|$. From (45)-(47), $|\mathcal{W}'_{U,h,n_1}| = |\mathcal{W}'_{U,v,n_1}| = |\mathcal{W}'_{U,d,n_1}| = M[(M-1)-3n_1]$ implying that $|\mathcal{W}'_{U,h}| = |\mathcal{W}'_{U,v}| = |\mathcal{W}'_{U,d}| = M\sum_{n_1=2}^{k-1}[(M-1)-3n_1] = M\left[(M-1)(k-2)-3\left(\frac{k(k-1)}{2}-1\right)\right]$. Therefore, $|\mathcal{W}'_{U,h}| + |\mathcal{W}'_{U,v}| + |\mathcal{W}'_{U,d}| = 3M\left[(M-1)(k-2)-3\left(\frac{k(k-1)}{2}-1\right)\right]$ where it can be easily proven that this number is equal to $|\mathcal{W}'_U|$ given in (43) in the cases $M=3k$, $M=3k+1$ and $M=3k-1$.

From (24) and (45): $W'^{(1)}_{U,h,n_1} = \left\{\pi^{m+n_2+1}(1,\pi^{-n_2-1}(1),\pi^{n_1-n_2-1}(1))\ ;\ n_2 = 2n_1\cdots M-2-n_1\right\}$ for $n_1 = 2,\ldots,k-1$. Performing the change of variable $N_2 = -n_2-1$ results in $N_2 = -2n_1-1 \equiv (M-1)-2n_1$ for $n_2 = 2n_1$ and $N_2 = n_1-M+1 \equiv n_1+1$ for $n_2 = (M-2)-n_1$. Therefore, $W'^{(1)}_{U,h,n_1} = \left\{\pi^{m-N_2}(1,\pi^{N_2}(1),\pi^{n_1+N_2}(1))\ ;\ N_2 = n_1+1,\ldots,(M-1)-2n_1\right\}$ which is equal to W'_{U,d,n_1} defined in (47) ($\{m\ ;\ m=0,\ldots,M-1\} \equiv \{m-N_2\ ;\ m=0,\ldots,M-1\}\ \forall\ N_2$).

From (24) and (45): $W'^{(2)}_{U,h,n_1} = \left\{\pi^{m+n_1+1}(1,\pi^{n_2-n_1}(1),\pi^{-n_1-1}(1))\ ;\ n_2 = 2n_1\cdots M-2-n_1\right\}$ for $n_1 = 2,\ldots,k-1$. Performing the change of variable $N_2 = n_2-n_1$ results in $N_2 = n_1$ for $n_2 = 2n_1$ and $N_2 = (M-2)-2n_1$ for $n_2 = (M-2)-n_1$. Therefore, $W'^{(2)}_{U,h,n_1} = \left\{\pi^{m+n_1+1}(1,\pi^{N_2}(1),\pi^{-n_1-1}(1))\ ;\ N_2 = n_1,\ldots,(M-2)-2n_1\right\}$ which is equal to W'_{U,v,n_1} defined in (46) ($\{m\ ;\ m=0,\ldots,M-1\} \equiv \{m+n_1+1\ ;\ m=0,\ldots,M-1\}\ \forall\ n_1$).

Appendix G

It can be observed that the set $\{\mathbf{s}^{(i)}\ ;\ i=1,\ldots,11\}$ remains unchanged if $\mathbf{s}$ is replaced by $\mathbf{s}^{(1)}$ or $\mathbf{s}^{(2)}$. Therefore, $\mathbf{s}$, $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$ are equivalent since they generate the same set of conjugates. Therefore, without loss of generality, the construction can be limited to $\mathcal{W}'_{U,h} \cup \mathcal{W}'_{L,d}$ where for every element of $\mathcal{W}'_{U,h}$ (resp. $\mathcal{W}'_{L,d}$), there are two elements in $\mathcal{W}'_{U,v} \cup \mathcal{W}'_{U,d}$ (resp. $\mathcal{W}'_{L,h} \cup \mathcal{W}'_{L,v}$) that share the same set of conjugates. In other words, for any codebook selected from $\mathcal{W}'_{U,h} \cup \mathcal{W}'_{L,d}$, there exists a number of equivalent codebooks where any codeword can be replaced by its first or second conjugate without altering the sets of spanned conjugates.

Observation *9:* Consider an element $\mathbf{s}$ of $\mathcal{W}'_{L,d}$, then either $\mathbf{s}^{(3)}$ or $\mathbf{s}^{(4)}$ or $\mathbf{s}^{(5)}$ fall in $\mathcal{W}'_{U,h}$.

Proof: Consider the third conjugate $\mathbf{s}^{(3)}$. Since $\mathbf{s}^{(3)} \in \mathcal{W}'_U$ for $\mathbf{s} \in \mathcal{W}'_L$ from (39), then $\mathbf{s}^{(3)}$ falls in either $\mathcal{W}'_{U,h}$ or $\mathcal{W}'_{U,v}$ or $\mathcal{W}'_{U,d}$. (i): If $\mathbf{s}^{(3)} \in \mathcal{W}'_{U,h}$, then the proof is complete. (ii): If $\mathbf{s}^{(3)} \in \mathcal{W}'_{U,v}$, then $[\mathbf{s}^{(3)}]^{(1)} = \mathbf{s}^{(5)} \in \mathcal{W}'^{(1)}_{U,v} = \mathcal{W}'_{U,h}$ where the last equality follows from observation 6. (iii): If $\mathbf{s}^{(3)} \in \mathcal{W}'_{U,d}$, then $[\mathbf{s}^{(3)}]^{(2)} = \mathbf{s}^{(4)} \in \mathcal{W}'^{(2)}_{U,d} = \mathcal{W}'_{U,h}$ where the last equality follows from observation 6. Therefore, in all cases, either $\mathbf{s}^{(3)}$ or $\mathbf{s}^{(4)}$ or $\mathbf{s}^{(5)}$ belong to $\mathcal{W}'_{U,h}$. ∎

Proposition *9:* An element of $\mathcal{W}'_{U,h}$ can be at most the third, fourth or fifth conjugate of one element in $\mathcal{W}'_{L,d}$.

Proof: Consider two distinct (modulo-M) elements $\mathbf{s} = \pi^m(1,\pi^{n_1}(1),\pi^{n_2}(1))$ and $\mathbf{s}' = \pi^m\left(1,\pi^{n'_1}(1),\pi^{n'_2}(1)\right)$ of $\mathcal{W}'_{L,d}$. We will next prove that the relation $\mathbf{s}^{(i)} = \mathbf{s}'^{(i')}$ can not hold for $(i,i') \in \{3,4,5\}^2$. The following cases need to be considered.

(i): $i=i'$: $\mathbf{s}^{(i)} = \mathbf{s}'^{(i)} \Rightarrow \mathbf{s} = \mathbf{s}'$ contradicting the fact that $\mathbf{s}$ and $\mathbf{s}'$ are distinct.

(ii): $(i,i') = (4,3)$: $\mathbf{s}'^{(3)} = \mathbf{s}^{(4)} \Rightarrow \mathbf{s}' = [\mathbf{s}^{(4)}]^{(8)}$ from (26). From (24), this relation results in:

$$\begin{aligned}\mathbf{s}' &= \pi^m\left(1,\pi^{-n_2}(1),\pi^{n_1-n_2-1}(1)\right)\\ &= \pi^m\left(1,\pi^{-(n_2-1)-1}(1),\pi^{(n_1-1)-(n_2-1)-1}(1)\right)\\ &= \left[\pi^m\left(1,\pi^{n_1-1}(1),\pi^{n_2-1}(1)\right)\right]^{(1)} \triangleq \mathbf{s}''^{(1)},\end{aligned} \tag{59}$$

where, following from the structure of $\mathcal{W}'_{L,d}$ in (47), the following cases arise. (ii.1): $n_2 \in \{n_1+2,\ldots,(M-2)-2n_1\}$. In this case, $\mathbf{s}'' \in \mathcal{W}'_{L,d}$ implying that $\mathbf{s}''^{(1)} \in \mathcal{W}'^{(1)}_{L,d} = \mathcal{W}'_{L,v}$ from observation 7. (ii.2): $n_2 = n_1+1$. In this case, $\mathbf{s}'' \in \mathcal{W}'_{L,v}$ implying that $\mathbf{s}''^{(1)} \in \mathcal{W}'^{(1)}_{L,v} = \mathcal{W}'_{L,h}$ from observation 7. Therefore, in both cases, $\mathbf{s}' = \mathbf{s}''^{(1)}$ can not belong to $\mathcal{W}'_{L,d}$.

(iii): $(i,i') = (5,4)$: $\mathbf{s}'^{(4)} = \mathbf{s}^{(5)} \Rightarrow \mathbf{s}' = [\mathbf{s}^{(5)}]^{(4)} = \pi^m\left(1,\pi^{-n_2}(1),\pi^{n_1-n_2-1}(1)\right)$ following from (24) and (26). This relation is the same as (59) resulting in the same findings as in case (ii).

(iv): $(i,i') = (5,3)$: $\mathbf{s}^{(5)} = \mathbf{s}'^{(3)} \Rightarrow \mathbf{s} = [\mathbf{s}'^{(3)}]^{(9)} = \pi^{m'}\left(1,\pi^{-n'_2}(1),\pi^{n'_1-n'_2-1}(1)\right)$ following from (24) and (26). This relation is equivalent to (59) where n_1 and n_2 need to be replaced by n'_1 and n'_2, respectively. Therefore, by interchanging the roles of $\mathbf{s}$ and $\mathbf{s}'$, the same conclusion that $\mathbf{s} \notin \mathcal{W}'_{L,d}$ can be drawn.

(v): The remaining cases $(i,i') \in \{(3,4),(4,5),(3,5)\}$ follow by interchanging the roles of $\mathbf{s}$ and $\mathbf{s}'$ in cases (ii), (iii) and (iv), respectively. ∎

Let $N_U \triangleq |\mathcal{C}_\mathcal{W} \cap \mathcal{W}'_{U,h}|$ and $N_L \triangleq |\mathcal{C}_\mathcal{W} \cap \mathcal{W}'_{L,d}|$ with $|\mathcal{C}_\mathcal{W}| = N_U + N_L$. Following from observation 9 and proposition 9, no two or more elements of $\mathcal{C}_\mathcal{W} \cap \mathcal{W}'_{L,d}$ can share the same third, fourth or fifth conjugate in $\mathcal{W}'_{U,h}$. Moreover, these conjugates must fall outside $\mathcal{C}_\mathcal{W}$ (i.e. inside $\mathcal{W}'_{U,h}\backslash\mathcal{C}_\mathcal{W}$) for $\mathcal{C}_\mathcal{W}$ to constitute a feasible solution. Consequently, $|\mathcal{W}'_{U,h}\backslash\mathcal{C}_\mathcal{W}| \geq |\mathcal{C}_\mathcal{W} \cap \mathcal{W}'_{L,d}|$ implying that $|\mathcal{W}'_{U,h}| - N_U \geq N_L$ resulting in $N_U + N_L = |\mathcal{C}_\mathcal{W}| \leq |\mathcal{W}'_{U,h}|$ where $|\mathcal{W}'_{U,h}|$ is given in (48).

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