On the Optimality of the Selection Transmit Diversity for MIMO-FSO Links with Feedback

Chadi Abou-Rjeily, *Member IEEE*

***Abstract*— We propose an optimal power allocation strategy for Multiple-Input-Multiple-Output (MIMO) Free-Space Optical (FSO) links with Intensity Modulation (IM) and Direct Detection (DD). The optimization is performed for shot noise limited systems in the presence of complete feedback. The derived analytical solution turns out to be the same as the selection transmit diversity scheme proposed in [1] for MISO-FSO systems corrupted by Gaussian noise. We also propose and analyze a novel transmission strategy for the limited-feedback case.**

***Index Terms*— Free-space optics, transmit diversity, power allocation, atmospheric turbulence.**

I. Introduction and Problem Formulation

Free-Space Optical (FSO) links often suffer from fading (or scintillation) that results from the variation of the index of refraction due to inhomogeneities in temperature and pressure changes. In this context, there is a growing interest in Multiple-Input-Multiple-Output (MIMO) techniques as a means of combating fading and leveraging the performance of FSO links [1]–[5]. In addition to fading, FSO links suffer from the following impairments: (i) shot noise that results from the random nature of the photo-generation process at the receiver and (ii) background noise, thermal noise and dark currents whose contributions are often modeled by an additive white Gaussian noise (AWGN). An accurate performance analysis must take the above two factors into consideration; however, this approach results in intractable results that do not offer clear insights on the performance of FSO systems. Consequently, simplifying assumptions are often made.

For example, in [2], AWGN was ignored and a "repetition coding" (RC) scheme was proposed for systems that suffer from shot noise (as well as background noise having Poisson statistics). This scheme corresponds to transmitting the same symbol from all optical sources with a uniform power allocation. RC was also analyzed in [3] for systems corrupted uniquely by AWGN. In [4], [5], non-negative totally-real space-time codes (STC) were proposed for FSO systems corrupted by Gaussian noise. In [1], a selection transmit diversity (STD) scheme was proposed for Multiple-Input-Single-Output (MISO) FSO links in the presence of AWGN while ignoring the shot noise. In the presence of a complete feedback, STD is based on the selection of the optical path having the highest scintillation. It was shown that STD achieves higher performance levels than RC and STC that do not require any feedback from the receiver to the transmitter.

The author is with the Department of Electrical and Computer Engineering of the Lebanese American University (LAU), Byblos, Lebanon. (e-mail: chadi.abourjeily@lau.edu.lb).

In this paper, we tackle the problem of transmit diversity for FSO links with feedback as a power allocation problem. In this context, the main contribution of our work corresponds to proving the optimality of the STD scheme as an optimal power allocation strategy for shot noise limited MIMO-FSO systems. In our work, AWGN is assumed to be negligible compared to the shot noise. This assumption is justified by the fact that transmit diversity schemes that are designed to combat fading result in the highest performance gains at high signal-to-noise ratios. Consequently, for these high performance systems where the received signal strength is sufficiently large, the signal-dependent shot noise and fading become the main limiting factors [2]. Other minor contributions are as follows: (i) we extend the STD scheme that was initially considered with MISO links to MIMO links and (ii) we propose an extension of the STD scheme to situations where a feedback with limited number of bits is available. We also prove the optimality of STD under the scenario considered in [1]; that is, MISO systems with AWGN in the absence of shot noise.

II. Optimal Power Allocation Strategy

Consider a MIMO-FSO link where M laser sources illuminate simultaneously an array of N distant photodetectors (PDs). Denote by P_m the fraction of the total transmit power allocated to the m-th laser for $m = 1, \dots, M$. For Q-ary PPM with RC, the conditional symbol error probability in the absence of background radiation can be written as [2]:

$$P_{e|A} = \frac{Q-1}{Q} e^{-\eta_s \sum_{n=1}^{N} \sum_{m=1}^{M} P_m a_{n,m}^2} \tag{1}$$

where $a_{n,m}$ stands for the path gain between the m-th laser and n-th PD and the channel state A is determined from the MN values taken by $a_{n,m}$ for $n = 1, \dots, N$ and $m = 1, \dots, M$. In this work, we adopt the lognormal turbulence-induced fading channel model [2]. η_s corresponds to the average number of photoelectrons per PPM slot due to the incident light signal:

$$\eta_s = \eta \frac{\mathcal{P}_r T_s / Q}{hf} = \eta \frac{E_s}{hf} \tag{2}$$

where η is the detector's quantum efficiency assumed to be equal to 1 in what follows, $h = 6.6 \times 10^{-34}$ is Planck's constant and f is the optical center frequency taken to be 1.94×10^{14} Hz (corresponding to a wavelength of 1550 nm). T_s stands for the symbol duration and $\mathcal{P}_r$ stands for the optical power that is incident on the receiver. Finally, $E_s \triangleq \mathcal{P}_r T_s / Q$ corresponds to the received optical energy per symbol.

Proposition: The optimal values of $\{P_m\}_{m=1}^{M}$ that minimize $P_{e|A}$ subject to the constraints $\sum_{m=1}^{M} P_m = 1$ and $P_m \geq 0$

for $m = 1, \dots, M$ are given by:

$$P_m = \delta_{m,\tilde{m}} \quad ; \quad \tilde{m} = \arg \max_{m=1,\dots,M} \{\sum_{n=1}^{N} a_{n,m}^2\} \tag{3}$$

where $\delta_{i,j} = 1$ for $i = j$ and $\delta_{i,j} = 0$ for $i \neq j$.

Proof: The proof is provided in the appendix.

Note that for the AWGN-limited MISO systems considered in [1] (in the absence of shot noise), the conditional probability of error can be written as: $P'_{e|A} = Q\left(\sqrt{K}(\sum_{m=1}^{M} P_m a_{1,m})^2\right)$ where K is a constant proportional to the SNR. Given that the solution proposed in (3) minimizes the error probability given in (1), then it maximizes the term $\sum_{m=1}^{M} P_m a_{1,m}^2$. Since this term is maximized for $P_m = \delta_{m,\tilde{m}}$ where $\tilde{m} = \arg\max\{a_{1,m}^2\}$ and since this selection procedure is equivalent to $\tilde{m} = \arg\max\{a_{1,m}\}$, then the choice $P_m = \delta_{m,\tilde{m}}$ also maximizes the term $\sum_{m=1}^{M} P_m a_{1,m}$ implying that it minimizes $P'_{e|A}$. Consequently, the proposed power allocation strategy is also optimal under the scenario considered in [1].

III. Feedback with limited number of bits

The STD scheme considered in the previous section requires $\lceil \log_2 M \rceil$ bits of feedback for indicating the laser from which the total optical power must be transmitted (the function $\lceil x \rceil$ rounds the real number x to the smallest integer that is greater than or equal to x). Assume now that the transmitter acquires a partial knowledge of the channel state information (CSI) via $b_f < \lceil \log_2 M \rceil$ feedback bits. For this scenario, the strategy that we propose consists of partitioning the set of M lasers into $K \triangleq 2^{b_f}$ disjoint groups. Based on the value taken by the b_f feedback bits, the transmitter selects one of the K groups and distributes the total optical power evenly among the elements of this group. Based on this formulation, the RC scheme of [2] is equivalent to the no-feedback case ($K = 1$ group having M elements) and the optical power is evenly distributed among the M lasers. In the same way, the STD scheme of [1] is equivalent to the complete-feedback case ($K = M$ groups having one element each) and only one laser is turned on based on the specific channel realization.

Denote by n_k the number of lasers in the k-th group for $k = 1, \dots, K$. The proposed strategy corresponds to deciding in favor of group $\tilde{k}$ and transmitting a normalized power of $\frac{1}{n_{\tilde{k}}}$ from each laser of this group if the integer $\tilde{k}$ is chosen as:

$$\tilde{k} = \arg \max_{k=1,\dots,K} A_k \triangleq \arg \max_{k=1,\dots,K} \frac{1}{n_k} \sum_{n=1}^{N} \sum_{j=1}^{n_k} a_{n,f(k,j)}^2 \tag{4}$$

where $f(k,j) \triangleq [\sum_{i=1}^{k-1} n_i] + j$ corresponds to the index of the j-th laser in the k-th group.

In this work, we adopt the lognormal fading model [2] where each path gain is modeled as a lognormal random variable (r.v.) with parameters μ and σ. These parameters satisfy the relation $\mu = -\sigma^2$ so that the mean path intensity is normalized to unity. The degree of fading is measured by the scintillation index defined as: S.I. $= e^{4\sigma^2} - 1$ [2].

The r.v. A_k defined in (4) corresponds to the summation of Nn_k lognormal r.v.s. Even though this lognormal-sum distribution is not known in closed-form, it is often approximated

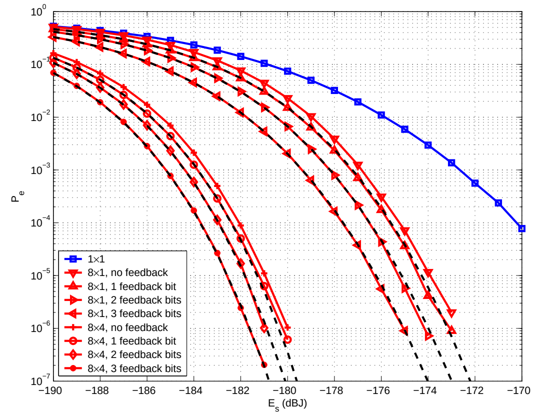

Fig. 1. Performance of 4-PPM over lognormal fading channels (S.I. = 0.6). Dashed lines correspond to the semi-analytical results obtained from (5).

by another lognormal distribution by a number of methods [6]. One of these methods that we will adopt in what follows is the Wilkinson's method [6]. In other words, for $Nn_k \neq 1$, A_k will be approximated by a lognormal r.v. whose parameters (that will be denoted by μ_{Nn_k} and σ_{Nn_k}) are determined from Wilkinson's method. Note that for $N = 1$ and $n_k = 1$, A_k is a lognormal r.v. with parameters $\mu_1 = 2\mu$ and $\sigma_1 = 2\sigma$.

Denote by $f_i(.)$ and $F_i(.)$ the probability density function (pdf) and cumulative distribution function (cdf) of the lognormal r.v. with parameters μ_i and σ_i, respectively. In this case, the cdf of the maximum gain $A_{\tilde{k}}$ can be written as: $F_{\max}(a) = \prod_{k=1}^{K} F_{Nn_k}(a)$ implying that the pdf of $A_{\tilde{k}}$ can be written as: $f_{\max}(a) = \sum_{k'=1}^{K} f_{Nn_{k'}}(a) \prod_{k' \neq k=1}^{K} F_{Nn_k}(a)$. For example, assume that $M = 8$, $N = 1$ and $b_f = 2$ (resulting in $K = 4$). If we partition the 8 lasers into 4 groups such that $n_1 = 1$, $n_2 = 2$, $n_3 = 2$ and $n_4 = 3$, then in this case: $F_{\max}(a) = F_1(a)F_2^2(a)F_3(a)$ resulting in $f_{\max}(a) = f_1(a)F_2^2(a)F_3(a) + 2f_2(a)F_1(a)F_2(a)F_3(a) + F_1(a)F_2^2(a)f_3(a)$. Note that for the MISO-STD scheme with complete feedback: $N = 1$ and $K = M$ implying that $n_k = 1$ for $k = 1, \dots, M$ resulting in $F_{\max}(a) = F_1^M(a)$ and $f_{\max}(a) = MF_1^{M-1}(a)f_1(a)$ which corresponds to the result obtained in [1].

Finally, the probability of error can be determined from:

$$P_e = \frac{Q-1}{Q} \int_0^{+\infty} e^{-\eta_s a^2} f_{\max}(a) \mathrm{d}a \tag{5}$$

Note that because of the complex expression taken by $f_{\max}(a)$, the above integral can not be solved analytically. Finally, since all path gains are identically distributed, then the choice of the specific lasers to be placed within the same group does not have any impact on the value taken by P_e.

Evaluating (5) under different scenarios showed the following main observation: for a given value of b_k, all possible partitions resulted in approximately the same performance level. In other words, there is no evident preference of one value of $(n_1, \dots, n_K)$ over another. For example, for $M = 7$ and $b_f = 1$, the possible values of (n_1, n_2) are $(1, 6)$, $(2, 5)$ and $(3, 4)$. In this case, plotting the result in (5) showed that the three corresponding error curves were extremely close to each other. This might follow from the independence of the path gains that add up coherently at the receiver side.

IV. Numerical Results

Fig. 1 shows the performance of 4-PPM with $M=8$ laser sources over lognormal fading channels having a scintillation index of 0.6. Results show the impact of the number of feedback bits on the performance. The proposed strategy in the limited-feedback case is capable of achieving different levels of performance that fall within that of the STD scheme [1] (complete feedback) and the RC scheme [2] (no feedback). Results also show the close match between simulations and the semi-analytical expression of P_e given in (5). Similar results were obtained in the presence of background radiation but were not presented here because of space limitations.

V. Conclusion

MIMO-FSO links with feedback do not require knowledge of the exact values taken by the path gains. These systems require solely the implementation of a certain sorting function at the receiver while perfect CSI can be realized via a finite number of feedback bits. While such a sorting function was provided in [1] in the case of complete feedback, we proved its optimality and extended it to systems with limited feedback.

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Appendix

From (1), minimizing $P_{e|A}$ is equivalent to minimizing:

$$e^{-\eta_s \sum_{m=1}^{M} P_m \sum_{n=1}^{N} a_{n,m}^2} = e^{-\sum_{m=1}^{M} k_m P_m} \tag{6}$$

where $k_m \triangleq \eta_s \sum_{n=1}^{N} a_{n,m}^2$ for $m=1,\dots,M$. Now, minimizing (6) is equivalent to minimizing the function:

$$f(P_1,\dots,P_M) = -\sum_{m=1}^{M} k_m P_m \tag{7}$$

In order to take the equality constraint $\sum_{m=1}^{M} P_m = 1$ and the M inequality constraints $-P_m \leq 0$ (for $m=1,\dots,M$) into consideration, we construct the Lagrangian function:

$$\Lambda(\underline{\mathbf{P}},\lambda,\underline{\mu}) = -\sum_{m=1}^{M} k_m P_m + \lambda(\sum_{m=1}^{M} P_m - 1) - \sum_{m=1}^{M} \mu_m P_m \tag{8}$$

At the point $(P_1,\dots,P_M)$ minimizing $f(P_1,\dots,P_M)$, the function Λ must satisfy the following M equations:

$$\frac{\partial \Lambda}{\partial P_m} = -k_m + \lambda - \mu_m = 0 \ ; \ m=1,\dots,M \tag{9}$$

Moreover, the differentiation of the function Λ with respect to the Lagrange multiplier λ implies that:

$$\frac{\partial \Lambda}{\partial \lambda} = \sum_{m=1}^{M} P_m - 1 = 0 \tag{10}$$

The Karush-Kuhn-Tucker (KKT) conditions that are necessary for the optimality of the final solution can be summarized in a set of M equalities and M inequalities as follows:

$$\mu_m P_m = 0 \ \ ; \ \ m=1,\dots,M \tag{11}$$

$$\mu_m \geq 0 \ \ ; \ \ m=1,\dots,M \tag{12}$$

Assume that $P_{\tilde{m}} > 0$ for a certain value of $\tilde{m} \in \{1,\dots,M\}$. Equation (11) implies that $\mu_{\tilde{m}} = 0$ resulting in $\lambda = k_{\tilde{m}}$ following from applying (9) for $m=\tilde{m}$. Replacing this value of λ in (9) for $m \neq \tilde{m}$ results in:

$$\mu_m = (k_{\tilde{m}} - k_m) \ \ ; \ \ m \neq \tilde{m} \tag{13}$$

Note that $\mu_m \neq 0$ for $m \neq \tilde{m}$ since the equality $k_{\tilde{m}} = k_m$ among the random variables k_m and $k_{\tilde{m}}$ that depend on the channel coefficients $\{a_{n,m}\}$ does not hold in general. Replacing $\mu_m \neq 0$ in (11) results in $P_m = 0$ for $m \neq \tilde{m}$ implying that $P_{\tilde{m}} = 1$ following from (10). On the other hand, the $M-1$ inequalities given in (12) hold if and only if $\mu_m > 0$ for $m \neq \tilde{m}$ resulting in:

$$k_m < k_{\tilde{m}} \ \ ; \ \ m \neq \tilde{m} \tag{14}$$

following from (13).

Therefore, the $2M+1$ equalities in equations (9)-(11) and the M inequalities in (12) will all hold and the solution $(P_1,\dots,P_M) = \mathbf{e_{\tilde{m}}}$ will be optimal if and only if (14) is satisfied (where $\mathbf{e_m}$ stands for the m-th row of the $M \times M$ identity matrix). Consequently, $\tilde{m}$ must be chosen as:

$$\tilde{m} = \arg\max_{m=1,\dots,M}\{k_m\} = \arg\max_{m=1,\dots,M}\{\sum_{n=1}^{N} a_{n,m}^2\} \tag{15}$$

resulting in the optimal solution proposed in (3).

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