Unipolar Space-Time Codes with Reduced Decoding Complexity for TH-UWB with PPM

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***Abstract*—In this paper, we consider the problem of Space-Time (ST) coding with unipolar Pulse Position Modulations (PPM) and propose a novel ST code that satisfies a large number of construction constraints rendering it superior to the existing PPM encoding schemes. In particular, the proposed 2×2 code achieves a full transmit diversity order while transmitting at a rate of 1 PPM-symbol per channel use. The proposed scheme can be associated with M-ary PPM constellations for all even values of M without introducing any constellation expansion. This renders the proposed scheme suitable for low cost carrier-less Ultra-Wideband (UWB) systems where information must be conveyed only by the time delays of the modulated sub-nanosecond pulses without introducing any amplitude amplifications or phase rotations. Finally, the proposed scheme can be associated with a reduced complexity optimal Maximum-Likelihood (ML) decoder that takes the structure of the proposed code into consideration in order to simplify the decoding procedure. We also propose a simple diversity-preserving suboptimal decoder that requires approximately half the number of multiplications compared to the ML decoder. Possible extensions to transmitters equipped with three antennas are also discussed in situations where a certain number of feedback bits is available.**

***Index Terms*—Time-Hopping Ultra-Wideband (TH-UWB), Space-Time (ST), Pulse Position Modulation (PPM).**

I. Introduction

There is a growing interest in applying Space-Time (ST) coding techniques on Time-Hopping Ultra-WideBand (TH-UWB) systems [1], [2]. For these systems, Pulse Position Modulation (PPM) is appealing since it is difficult to control the phase and amplitude of the very low duty-cycle sub-nanosecond pulses used to convey the information symbols.

Two different approaches can be adopted for the construction of ST codes suitable for PPM. The first approach consists of applying one of the numerous ST codes proposed in the literature for QAM, PAM or PSK [3], [4]. In this context, it can be easily proven that these codes remain fully diverse with PPM [2]. However, the disadvantage is that all of these codes introduce phase rotations or amplitude amplifications in order to achieve a full transmit diversity order and, consequently, they introduce an additional constellation expansion when associated with PPM. For example, while single-antenna PPM systems transmit unipolar pulses, applying the Alamouti code [3] with PPM necessitates the transmission of pulses having positive and negative polarities.

In order to overcome the above disadvantage, the second approach consists of constructing shape-preserving PPM-specific unipolar codes [5]–[7]. However, all of these codes are exclusive to binary PPM (or OOK) and they permit to achieve a full transmit diversity order because of the structure of such binary constellations that are composed of a signal and its opposite defined as the signal obtained by reversing the roles of "on" and "off" [5]. Various extensions to M-ary constellations were proposed in [8]. However, this was realized at the expense of an increased receiver complexity since, to preserve diversity, these codes must be associated with complex nonlinear decoders such as the sphere decoder [9].

The first contribution of this paper is the proposition of a rate-1, fully diverse and shape-preserving ST block code for unipolar PPM with two transmit antennas. The advantage over [5]–[7] is that the proposed scheme can be associated with M-PPM for all even values of M. The advantage over [8] is that the proposed code admits a reduced complexity maximum-likelihood decodability. Note that unlike [3], this simplified decodability is realized even though the proposed scheme is unipolar and not orthogonal. Note that symbol-by-symbol decodable codes that are not based on the orthogonal design were first proposed in [10] for QAM constellations.

Inspired from [11], the second contribution consists of extending the proposed scheme to three antennas when 1, 2 or 3 feedback bits are available. In this case, a transmit diversity order of three can be achieved with a simple reduced complexity decodability. Note that in the absence of feedback, the only existing solution for three-antenna systems is exclusive to M-PPM with $M=3$ or $M\geq 5$ [8]. Finally, we propose simple suboptimal decoders that can take advantage from the transmit diversity offered by the proposed schemes. The proposed optimal and suboptimal decoders are compared in terms of performance and complexity that is measured by the number of multiplications necessary for decoding one information symbol.

II. Two transmit antennas with no feedback

A. System Model

Consider a TH-UWB system where the transmitter is equipped with 2 antennas and the receiver is equipped with Q antennas. In what follows, we propose a minimal-delay diversity scheme that extends over two symbol durations. Denote by $s_p(t)$ the signal transmitted from the p-th antenna for $p = 1, 2$. We propose the following structure for the transmitted signals:

$$s_1(t) = w\left(t-(p_1-1)\delta\right) + w(t-T_s-(p_2-1)\delta) \quad (1)$$

$$s_2(t) = w(t-(\pi(p_2)-1)\delta) + w(t-T_s-(p_1-1)\delta) \quad (2)$$

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where $p_i \in \{1, \ldots, M\}$ corresponds to the modulation position of the i-th information symbol for $i = 1, 2$. $w(t)$ is the pulse waveform of duration T_w normalized to have an energy of $E_s/2$ where E_s is the energy used to transmit one information symbol and the normalization by 2 insures the same transmission level as in the single-antenna case. The modulation delay δ corresponds to the separation between two consecutive PPM positions while T_s stands for the symbol duration.

Note that two M-PPM symbols are transmitted during two symbol durations and the proposed scheme transmits at a rate of one symbol per channel use (PCU). No reference to the TH sequence was made since all antennas of the same user are supposed to share the same TH sequence resulting in the same average multi-user interference as in the single-antenna case.

The permutation function $\pi(.)$ in eq. (2) is defined by:

$$\pi(m) = (m \mod 2) + 2\left\lfloor \frac{m-1}{2} \right\rfloor + 1 \quad (3)$$

where $\lfloor x \rfloor$ rounds the real number x to the nearest integer that is less than or equal to it.

From eq. (1) and eq. (2), the pulses transmitted from the two antennas during two consecutive symbol durations occupy the positions p_1, p_2 and $\pi(p_2)$. Since $\pi(.)$ defines a mapping over the elements of the set $\{1, \ldots, M\}$ when M is even, then $p_1, p_2, \pi(p_2) \in \{1, \ldots, M\}$. Moreover, the transmission strategy described in eq. (1) and eq. (2) does not introduce any amplitude scaling. Consequently, during each symbol duration, only one unipolar pulse occupying one out of M possible positions is transmitted. Therefore, the proposed scheme does not introduce any expansion to the M-PPM constellation for all values of M.

In what follows, M is limited to take even values. M-ary PPM constellations are M-dimensional constellations where the information symbols are represented by M-dimensional vectors that belong to the following signal set:

$$\mathcal{C} = \{I_{M,m} \; ; \; m = 1, \ldots, M\} \quad (4)$$

where $I_{M,m}$ is the m-th column of the $M \times M$ identity matrix I_M.

Designate by $a_i \triangleq [a_{i,1} \; \cdots \; a_{i,M}]^T = I_{M,p_i} \in \mathcal{C}$ the M-dimensional vector representation of the i-th information symbol for $i = 1, 2$. Equations (1) and (2) can be written as:

$$s_1(t) = \sum_{m=1}^{M} [a_{1,m} w(t-(m-1)\delta) + a_{2,m} w(t-T_s-(m-1)\delta)] \quad (5)$$

$$s_2(t) = \sum_{m=1}^{M} \left[a_{2,\pi(m)} w(t-(m-1)\delta) + a_{1,m} w(t-T_s-(m-1)\delta)\right] \quad (6)$$

where $a_{i,m}$ is the m-th component of a_i with $a_{i,m} = 1$ if $m = p_i$ and $a_{i,m} = 0$ otherwise.

The received signal at the q-th antenna can be written as:

$$r_q(t) = \sum_{p=1}^{2} s_p(t) * g_{q,p}(t) + n_q(t) \quad (7)$$

where $*$ stands for convolution and $n_q(t)$ is the noise at the q-th antenna which is supposed to be real AWGN with double sided spectral density $N_0/2$. $g_{q,p}(t)$ stands for the impulse response of the frequency selective channel between the p-th transmit antenna and the q-th receive antenna.

In order to take advantage from the rich multi-path diversity of the UWB channels, a L-th order Rake is used after each receive antenna. Designate by $y_{q,l,i,m}$ the decision variable collected at the l-th Rake finger of the q-th receive antenna during the m-th position of the i-th symbol duration for $q = 1, \ldots, Q$, $l = 1, \ldots, L$, $i = 1, 2$ and $m = 1, \ldots, M$. Each one of these $2QLM$ decision variables is given by:

$$y_{q,l,i,m} = \int_{-\infty}^{+\infty} r_q(t) w(t - \Delta_l - (i-1)T_s - (m-1)\delta) \mathrm{d}t \quad (8)$$

where $\Delta_l \triangleq (l-1)T_w$ stands for the delay of the l-th finger of the Rake.

Designate by T_c the delay spread of the UWB channel ($T_c \gg T_w$). Inter-Symbol-Interference (ISI) can be eliminated by choosing $T_s \geq T_c + T_w$. In the same way, the received PPM constellation is orthogonal if the modulation delay satisfies the relation: $\delta \geq T_c + T_w$. In what follows, we consider orthogonal received PPM constellations in the absence of ISI since only in this case the proposed scheme can be associated with a ML decoder having a reduced complexity. In this case, the decision variables given in eq. (8) can be written as:

$$y_{q,l,1,m} = h_{q,1,l} a_{1,m} + h_{q,2,l} a_{2,\pi(m)} + n_{q,l,1,m} \quad (9)$$

$$y_{q,l,2,m} = h_{q,2,l} a_{1,m} + h_{q,1,l} a_{2,m} + n_{q,l,2,m} \quad (10)$$

where $n_{q,l,i,m}$ stands for the noise term during the i-th symbol duration: $n_{q,l,i,m} = \int_{-\infty}^{+\infty} n_q(t) w(t - \Delta_l - (i-1)T_s - (m-1)\delta) \mathrm{d}t$. Moreover, since $\delta \geq 2T_w$ and $\Delta_l - \Delta_{l-1} \geq T_w$, it can be easily proven that these noise terms are white. In eq. (9) and eq. (10), the channel coefficients are given by: $h_{q,p,l} = \int_{-\infty}^{+\infty} h_{q,p}(t) w(t - \Delta_l) \mathrm{d}t$ where $h_{q,p}(t) \triangleq g_{q,p}(t) * w(t)$.

B. Optimal ML Decoding

1) Decoding Strategy: Despite the absence of orthogonality between the transmitted data streams (since the transmitted signals are unipolar), we propose a simple ML decoder that takes advantage from the structure of the proposed code in order to simplify the decoding procedure.

At a first time, we propose to partition the M PPM positions into $M/2$ slots containing 2 positions each. In this case, the n-th slot will contain positions $2n-1$ and $2n$ for $n = 1, \ldots, M/2$. Assume that the second symbol (whose position is given by p_2) is in the n-th slot: $p_2 \in \{2n-1, 2n\}$. This implies that $a_{2,m} = 0$ for $m \neq 2n-1$ and $m \neq 2n$. On the other hand, eq. (3) implies that $\pi(2n-1) = 2n$ and $\pi(2n) = 2n-1$. Moreover, one of the values in $\{a_{2,2n-1}, a_{2,2n}\}$ is equal to 1 while the other value is equal to 0. Consequently, conditioned on the presence of p_2 in the n-th slot, the components of vector a_2 satisfy the following relation:

$$a_{2,\pi(m)} = \begin{cases} 0, & m \notin \{2n-1, 2n\}; \\ -a_{2,m} + 1, & m \in \{2n-1, 2n\}. \end{cases} \quad (11)$$

Consequently, for $m \notin \{2n-1, 2n\}$, eq. (9) and eq. (10) can be written as:

$$y_{q,l,1,m} = h_{q,1,l} a_{1,m} + n_{q,l,1,m} \quad (12)$$
$$y_{q,l,2,m} = h_{q,2,l} a_{1,m} + n_{q,l,2,m} \quad (13)$$

While for $m \in \{2n-1, 2n\}$, eq. (9) and eq. (10) can be written as:

$$(y_{q,l,1,m} - h_{q,2,l}) = h_{q,1,l} a_{1,m} - h_{q,2,l} a_{2,m} + n_{q,l,1,m} \quad (14)$$
$$y_{q,l,2,m} = h_{q,2,l} a_{1,m} + h_{q,1,l} a_{2,m} + n_{q,l,2,m} \quad (15)$$

Equations (12)-(15) resemble the input-output relations of the Alamouti code [3]. Consequently, the first information symbol (which is represented by the vector a_1) can be decoded by constructing the following M decision variables $\{b_{1,m}\}_{m=1}^{M}$. For $m \notin \{2n-1, 2n\}$, $b_{1,m}$ is given by:

$$b_{1,m} = \sum_{q=1}^{Q} \sum_{l=1}^{L} [h_{q,1,l} y_{q,l,1,m} + h_{q,2,l} y_{q,l,2,m}] \quad (16)$$

while for $m \in \{2n-1, 2n\}$, $b_{1,m}$ takes the following form:

$$b_{1,m} = \sum_{q=1}^{Q} \sum_{l=1}^{L} [h_{q,1,l}(y_{q,l,1,m} - h_{q,2,l}) + h_{q,2,l} y_{q,l,2,m}] \quad (17)$$

implying that $b_{1,m}$ can be written as:

$$b_{1,m} \triangleq b_{1,m}^{(0)} + \begin{cases} 0, & m \notin \{2n-1, 2n\}; \\ K_1, & m \in \{2n-1, 2n\}. \end{cases} \quad (18)$$

where $b_{1,m}^{(0)} \triangleq \sum_{q=1}^{Q} \sum_{l=1}^{L} [h_{q,1,l} y_{q,l,1,m} + h_{q,2,l} y_{q,l,2,m}]$ depends on the transmitted signal and $K_1 \triangleq -\sum_{q=1}^{Q} \sum_{l=1}^{L} h_{q,1,l} h_{q,2,l}$ is constant (it depends only on the channel realization).

On the other hand, since we assume that a_2 is in the n-th slot, then for the detection of a_2 we need to construct the following two decision variables:

$$b_{2,m} = \sum_{q=1}^{Q} \sum_{l=1}^{L} [-h_{q,2,l}(y_{q,l,1,m} - h_{q,2,l}) + h_{q,1,l} y_{q,l,2,m}] \quad (19)$$
$$\triangleq b_{2,m}^{(0)} - b_{2,m}^{(1)} + K_2 \quad ; \quad m \in \{2n-1, 2n\} \quad (20)$$

where $b_{2,m}^{(0)} \triangleq \sum_{q=1}^{Q} \sum_{l=1}^{L} h_{q,1,l} y_{q,l,2,m}$, $b_{2,m}^{(1)} \triangleq \sum_{q=1}^{Q} \sum_{l=1}^{L} h_{q,2,l} y_{q,l,1,m}$ and $K_2 = \sum_{q=1}^{Q} \sum_{l=1}^{L} h_{q,2,l}^2$ is a positive constant.

Note that in the absence of noise, eq. (18) and eq. (20) imply that:

$$b_{i,m} = \begin{cases} \sum_{q=1}^{Q} \sum_{l=1}^{L} \left(h_{q,1,l}^2 + h_{q,2,l}^2\right), & m = p_i; \\ 0, & m \neq p_i. \end{cases} \quad ; \quad i = 1,2 \quad (21)$$

where eq. (21) holds in the case where the conditioning is made on the correct slot number n.

A detailed analysis of the diversity order of the proposed ST code will be given in section II-D. However, in a simplified manner, eq. (21) shows that at high signal-to-noise ratios $b_{i,p_i} \ll 1$ if and only if $|h_{q,p,l}| \ll 1$ for $q = 1, \dots, Q$, $p = 1,2$ and $l = 1, \dots, L$. In other words, the information symbols are lost only when the PQ sub-channels $g_{q,p}(t)$ suffer from fading over a duration LT_w. Therefore, the proposed scheme achieves full transmit, receive and multi-path diversities.

Since p_1 can occupy any one of the positions $\{1, \dots, M\}$, then conditioned on the event that a_2 is in the n-th slot, eq. (18) implies that p_1 can be decoded from:

$$\begin{aligned} \tilde{p}_1(n) &= \arg \max_{m=1,\dots,M} ([b_{1,1} \ \cdots \ b_{1,M}]) \\ &= \arg \max_{m=1,\dots,M} \left(\left[b_{1,1}^{(0)} \ \cdots \ b_{1,M}^{(0)} \right] + K_1 \left((I_{M/2,n})^T \otimes [1 \quad 1] \right) \right) \end{aligned} \quad (22)$$

where $I_{M/2,n}$ stands for the n-th column of the $M/2 \times M/2$ identity matrix $I_{M/2}$ and $\otimes$ stands for the Kronecker product.

On the other hand, given that p_2 can occupy only one of the two positions $\{2n-1, 2n\}$, then conditioned on the event that a_2 is in the n-th slot, eq. (20) implies that p_2 can be decoded from:

$$\begin{aligned} \tilde{p}_2(n) &= 2(n-1) + \arg\max ([b_{2,2n-1} \ , \ b_{2,2n}]) \\ &= 2(n-1) + \arg\max \left(\left[b_{2,2n-1}^{(0)}, b_{2,2n}^{(0)} \right] - \left[b_{2,2n-1}^{(1)}, b_{2,2n}^{(1)} \right] + K_2 \right) \\ &\equiv 2(n-1) + \arg\max \left(\left[b_{2,2n-1}^{(0)}, b_{2,2n}^{(0)} \right] - \left[b_{2,2n-1}^{(1)}, b_{2,2n}^{(1)} \right] \right) \end{aligned} \quad (23)$$

where the last equation follows since K_2 does not depend on the transmitted information symbols.

Let $\tilde{a}_1(n) \triangleq I_{M,\tilde{p}_1(n)}$ and $\tilde{a}_2(n) \triangleq I_{M,\tilde{p}_2(n)}$. After repeating the operations described in eq. (22) and eq. (23) $M/2$ times to construct the sets $\{\tilde{a}_1(n)\}_{n=1}^{M/2}$ and $\{\tilde{a}_2(n)\}_{n=1}^{M/2}$, the receiver decides in favor of $(\hat{a}_1, \hat{a}_2) = (\tilde{a}_1(\hat{n}), \tilde{a}_2(\hat{n}))$ where:

$$\begin{aligned} \hat{n} = \underset{n=1,\dots,\frac{M}{2}}{\text{argmin}} \sum_{q,l,m} \Big[& \left(y_{q,l,1,m} - h_{q,1,l} \tilde{a}_{1,m}(n) - h_{q,2,l} \tilde{a}_{2,\pi(m)}(n) \right)^2 \\ & + \left(y_{q,l,2,m} - h_{q,2,l} \tilde{a}_{1,m}(n) - h_{q,1,l} \tilde{a}_{2,m}(n) \right)^2 \Big] \end{aligned} \quad (24)$$

where $\tilde{a}_{i,m}(n)$ is the m-th component of the vector $\tilde{a}_i(n)$ for $i = 1,2$.

Because of the structure of the PPM constellation, the decision rule in eq. (24) can be significantly simplified. In the appendix we prove that an equivalent decision rule is given by:

$$\begin{aligned} \hat{n} = \arg \max_{n=1,\dots,M/2} \sum_{q=1}^{Q} \sum_{l=1}^{L} \Big[& h_{q,1,l} \left(y_{q,l,1,\tilde{p}_1(n)} + y_{q,l,2,\tilde{p}_2(n)} \right) \\ & + h_{q,2,l} \left(y_{q,l,2,\tilde{p}_1(n)} + y_{q,l,1,\pi(\tilde{p}_2(n))} \right) - h_{q,1,l} h_{q,2,l} \delta_{\lceil \tilde{p}_1(n)/2 \rceil, n} \Big] \end{aligned} \quad (25)$$

where $\delta_{i,j}$ stands for Kronecker's delta function ($\delta_{i,j} = 1$ for $i = j$ and $\delta_{i,j} = 0$ for $i \neq j$) and where $\lceil x \rceil$ rounds the real number x to the nearest integer that is greater than or equal to it.

Following from the definitions of the intermediate decision variables $b_{1,m}^{(0)}$, $b_{2,m}^{(0)}$ and $b_{2,m}^{(1)}$ and the constant K_1 given in eq. (18) and eq. (20), the decision rule in eq. (25) can be written as:

$$\hat{n} = \underset{n=1,\dots,\frac{M}{2}}{\text{argmax}} \left[b_{1,\tilde{p}_1(n)}^{(0)} + b_{2,\tilde{p}_2(n)}^{(0)} + b_{2,\pi(\tilde{p}_2(n))}^{(1)} + K_1 \delta_{\lceil \tilde{p}_1(n)/2 \rceil, n} \right] \quad (26)$$

2) Decoding Complexity: In this subsection, we evaluate the complexity of the proposed ML decoder in terms of the number of multiplications required for decoding a pair of information symbols (a_1, a_2). This simplified approach neglects the complexity of the additions and comparisons involved in the decoding algorithm.

The ML decoding procedure described in the previous subsection can be summarized in the following steps:

1) a) The receiver calculates the M intermediate decision variables $\{b_{1,m}^{(0)}\}_{m=1}^{M}$ described in eq. (18). This step necessitates $2QLM$ multiplications.
   b) The receiver calculates the M intermediate decision variables $\{b_{2,m}^{(0)}\}_{m=1}^{M}$ described in eq. (20). This step necessitates QLM multiplications.
   c) The receiver calculates the M intermediate decision variables $\{b_{2,m}^{(1)}\}_{m=1}^{M}$ described in eq. (20). This step necessitates QLM multiplications.
   d) The receiver calculates the constant K_1 in eq. (18). This step necessitates QL multiplications. Note that the constant K_2 in eq. (20) is not required for further decoding steps.
2) The receiver repeats the operation described in eq. (22) $M/2$ times to construct the set $\{\tilde{p}_1(n)\}_{n=1}^{M/2}$. This operation does not require any number of multiplications. In fact, the constant K_1 in eq. (22) is multiplied by either zero or one and then added to the decision vector to determine the maximum component of this vector. Evidently, there are no multiplications involved in this procedure.
3) The receiver repeats the operation described in eq. (23) $M/2$ times to construct the set $\{\tilde{p}_2(n)\}_{n=1}^{M/2}$. Once again, this procedure does not require any number of multiplications. In fact, the constants $\{2(n-1)\}_{n=1}^{M/2} = \{0, 2, \ldots, M-2\}$ can be calculated and stored before the decoding of each symbol pair.
4) The decoder evaluates the $M/2$ decision metrics in eq. (26) and decides in favor of the positions $(\tilde{p}_1(n), \tilde{p}_2(n))$ having the largest metric. No multiplications are involved in this step since evaluating the quantity

$$K_1 \delta_{\lceil \tilde{p}_1(n)/2 \rceil, n} = \begin{cases} K_1, & \tilde{p}_1(n) \in \{2n-1, 2n\}; \\ 0, & \tilde{p}_1(n) \notin \{2n-1, 2n\}. \end{cases} \quad (27)$$

does not require any multiplications.

Note that step (1) is performed before conditioning on the slot index n of the second information symbol. The remaining steps (2)-(4) perform the appropriate comparisons associated with the assumption that the second symbol is in the n-th slot.

From what preceded, the ML decoder requires $4QLM+QL$ multiplications for the detection of one pair of information symbols. Note also that the constant K_1 (that requires QL multiplications) depends only on the channel realization. Consequently, K_1 needs to be calculated once for each channel realization. Assuming a block fading channel that extends over N pairs of symbol durations, decoding the $2N$ information symbols requires $4NQLM + QL$ multiplications (rather than $N(4QLM + QL)$ multiplications). Note that since in IR-UWB systems the information on the phase is not retained, all of the above multiplications correspond to real-valued multiplications.

Note that the complexity of the proposed ML decoder is comparable to that of the Alamouti code [3]. In fact, for decoding N pairs of one-dimensional PAM symbols, the Alamouti code requires $4NQL$ real-valued multiplications. On the other hand, for decoding N pairs of M-dimensional M-PPM symbols, the proposed decoder requires $4NQLM + QL$ real-valued multiplications. In other words, the additional complexity of the proposed decoder follows mainly from the dimensionality of the PPM signal set and not from the structure of the encode/decoder.

C. Suboptimal Decoding

1) Decoding Strategy: In this section, we propose a suboptimal decoder that has a lower decoding complexity. The decoding procedure at the receiver side can be simplified by constructing the following QLM decision variables:

$$z_{q,l,i,n} = y_{q,l,i,2n-1} - y_{q,l,i,2n} \quad ; \quad n = 1, \ldots, M/2 \quad (28)$$

Based on the permutation rule given in eq. (3), it follows that $\pi(2n) = 2n-1$ and $\pi(2n-1) = 2n$ for $n = 1, \ldots, M/2$. Consequently, using eq. (9) and eq. (10) in eq. (28) results in:

$$z_{q,l,1,n} = h_{q,1,l} s_{1,n} - h_{q,2,l} s_{2,n} + n'_{q,l,1,n} \quad (29)$$
$$z_{q,l,2,n} = h_{q,2,l} s_{1,n} + h_{q,1,l} s_{2,n} + n'_{q,l,2,n} \quad (30)$$

where:

$$s_{i,n} \triangleq a_{i,2n-1} - a_{i,2n} \quad ; \quad i = 1, 2 \quad ; \quad n = 1, \ldots, M/2 \quad (31)$$

and $s_{i,n} \in \{0, \pm 1\}$ since $a_{i,1}, \ldots, a_{i,M} \in \{0, 1\}$ for $i = 1, 2$. In the same way, the noise terms are given by: $n'_{q,l,i,n} = n_{q,l,i,2n-1} - n_{q,l,i,2n}$ for $i = 1, 2$.

Note that the input-output relations given in eq. (29-30) are similar to the relations verified by the Alamouti code [3]. However, the orthogonal-like behavior of the proposed scheme is achieved mainly because of the position permutations described in eq. (3) and the associated decoding technique without necessitating any polarity inversion of the transmitted PPM pulses.

Finally, assuming perfect channel state information at the receiver side (knowledge of the coefficients $h_{q,p,l}$), the decisions taken on the information symbols will be based on the following M decision variables (for $n = 1, \ldots, M/2$):

$$Z_{1,n} = \sum_{q=1}^{Q} \sum_{l=1}^{L} \left[h_{q,1,l} z_{q,l,1,n} + h_{q,2,l} z_{q,l,2,n} \right] \quad (32)$$

$$Z_{2,n} = \sum_{q=1}^{Q} \sum_{l=1}^{L} \left[-h_{q,2,l} z_{q,l,1,n} + h_{q,1,l} z_{q,l,2,n} \right] \quad (33)$$

Replacing equations (29) and (30) in equations (32) and (33) results in (for $i = 1, 2$ and $n = 1, \ldots, M/2$):

$$Z_{i,n} = \left[\sum_{q=1}^{Q} \sum_{p=1}^{2} \sum_{l=1}^{L} h_{q,p,l}^2 \right] s_{i,n} + N_{i,n} \quad (34)$$

where $N_{1,n} = \sum_{q,l}\left[h_{q,1,l}n'_{q,l,1,n} + h_{q,2,l}n'_{q,l,2,n}\right]$ and $N_{2,n} = \sum_{q,l}\left[-h_{q,2,l}n'_{q,l,1,n} + h_{q,1,l}n'_{q,l,2,n}\right]$. It can be easily proven that these noise terms are still white.

In a simplified manner, by inspecting eq. (34) we observe that $Z_{i,n} \ll 1$ if and only if the $2QL$ channel coefficients $h_{q,p,l}$ (for $q = 1, \dots, Q$, $p = 1, 2$ and $l = 1, \dots, L$) all have small magnitudes. This shows that the overall diversity order of the system is $2QL$. Consequently, the proposed suboptimal decoder preserves the transmit diversity order of the proposed ST code.

Since the modified symbols $s_{1,n}$ and $s_{2,n}$ given in eq. (31) can be equal to zero, then the first step in decoding the i-th information symbol consists of calculating the integer $\hat{n}_i$ such that:

$$\hat{n}_i = \arg\max_{n=1,\dots,M/2} |Z_{i,n}| \quad ; \quad i = 1, 2 \tag{35}$$

Following from eq. (35), the reconstituted position of the i-th PPM information symbol is:

$$\hat{p}_i = \begin{cases} 2\hat{n}_i - 1, & Z_{i,\hat{n}_i} \geq 0; \\ 2\hat{n}_i, & Z_{i,\hat{n}_i} < 0. \end{cases} \quad ; \quad i = 1, 2 \tag{36}$$

in other words, the vector representation of the i-th reconstituted information symbol will be given by: $\hat{a}_i = I_{M,\hat{p}_i} \in \mathcal{C}$ for $i = 1, 2$ where $\mathcal{C}$ is given in eq. (4).

2) Decoding Complexity: The proposed suboptimal decoding procedure can be summarized in the following steps:

1) The receiver calculates the QLM intermediate decision variables $z_{q,l,i,n}$ given in eq. (28) for $q = 1, \dots, Q$, $l = 1, \dots, L$, $i = 1, 2$ and $n = 1, \dots, M/2$. This step does not require any number of multiplications.
2) The receiver calculates the $M/2$ decision variables $\{Z_{1,n}\}_{n=1}^{M/2}$ given in eq. (32). This step necessitates $2QL(M/2) = QLM$ multiplications.
3) The receiver calculates the $M/2$ decision variables $\{Z_{2,n}\}_{n=1}^{M/2}$ given in eq. (33). This step necessitates $2QL(M/2) = QLM$ multiplications.
4) The receiver decides in which slots are the two information symbols present according to eq. (35). This step does not require any number of multiplications.
5) The receiver decides in which positions within the slots (determined in the previous step) are the two information symbols present according to eq. (36). This step does not require any number of multiplications. In fact, if $Z_{i,\hat{n}_i}$ is positive (resp. negative), the receiver decides in favor of the first (resp. second) position withing slot $\hat{n}_i$ for $i = 1, 2$.

Consequently, for the detection of N pairs of information symbols, the simplified suboptimal decoder requires $2NQLM$ multiplications which is approximately half the number of multiplications required by the optimal ML decoder (for large values of N).

D. Diversity Order

In the previous sections, we realized heuristically that the proposed code profits from a full transmit diversity order (whether with the optimal or the suboptimal decoders). In this section, we adopt a more rigorous approach for proving that the proposed scheme is fully diverse based on the design criteria of [12]. Designate by $C(a_1, a_2)$ the $2M \times 2$ codeword whose $((p-1)M+m, i)$-th entry corresponds to the amplitude of the pulse (if any) transmitted at the m-th position of the p-th antenna during the i-th symbol duration for $p = 1, 2$, $m = 1, \dots, M$ and $i = 1, 2$. Based on eq. (5) and eq. (6), $C(a_1, a_2)$ can be written as:

$$C(a_1, a_2) = \begin{bmatrix} a_{1,1} & \cdots & a_{1,M} & a_{2,\pi(1)} & \cdots & a_{2,\pi(M)} \\ a_{2,1} & \cdots & a_{2,M} & a_{1,1} & \cdots & a_{1,M} \end{bmatrix}^T \tag{37}$$

Following from the linearity of the code and from [2], [12], the code is fully diverse if:

$$\text{rank}\left[C(a_1 - a'_1, a_2 - a'_2)\right] = 2 \quad \forall \quad (a_1, a_2) \neq (a'_1, a'_2) \tag{38}$$

where a_1, a'_1, a_2 and a'_2 belong to the set $\mathcal{C}$ given in eq. (4). Vectors $(a_1 - a'_1)$ and $(a_2 - a'_2)$ have the following structure: they can either be equal to the all-zero vectors or they can have one component that is equal to $+1$, one component that is equal to -1 and $M - 2$ zero components.

In what follows, $C(a_1 - a'_1, a_2 - a'_2)$ will be denoted by C when there is no ambiguity. On the other hand, rank$(C) < 2$ if there exists a nonzero real number k such that $C_2 = kC_1$ where C_i stands for the i-th column of C for $i = 1, 2$. Moreover, given that the elements of C belong to the set $\{0, \pm 1\}$, then $k = \pm 1$. Let n be an odd integer that belongs to $\{1, \dots, M\}$. Investigating the n-th and $(M+n)$-th rows of C respectively, the relation $C_2 = kC_1$ implies that:

$$a_{2,n} - a'_{2,n} = k(a_{1,n} - a'_{1,n}) \tag{39}$$
$$a_{1,n} - a'_{1,n} = k(a_{2,\pi(n)} - a'_{2,\pi(n)}) \tag{40}$$

Combining the last equations results in:

$$a_{2,n} - a'_{2,n} = k^2(a_{2,\pi(n)} - a'_{2,\pi(n)}) = a_{2,n+1} - a'_{2,n+1} \tag{41}$$

since $k^2 = 1$ and $\pi(n) = n + 1$ when $n \in \{1, \dots, M\}$ is odd.

Consequently, C is rank deficient if and only if:

$$a_{2,n} - a_{2,n+1} = a'_{2,n} - a'_{2,n+1} \; ; \; n \in \{1, \dots, M\} \text{ is odd} \tag{42}$$

Given that $(a_{2,n}, a_{2,n+1}) \in \{(0,0), (0,1), (1,0)\}$ and $(a'_{2,n}, a'_{2,n+1})$ belongs to the same set, then eq. (42) can be verified if and only if $a_{2,n} = a'_{2,n}$ and $a_{2,n+1} = a'_{2,n+1}$ for all odd integers n in $\{1, \dots, M\}$. Moreover, from eq. (39), $a_{2,n} = a'_{2,n}$ implies that $a_{1,n} = a'_{1,n}$. In the same way, from eq. (40), $a_{2,n+1} = a'_{2,n+1}$ implies that $a_{1,n+1} = a'_{1,n+1}$. Finally, $C(a_1 - a'_1, a_2 - a'_2)$ is rank deficient only when $a_1 = a'_1$ and $a_2 = a'_2$. Therefore, eq. (38) is verified and the code is fully diverse. Note that for non-orthogonal constellations eq. (9) and eq. (10) do not hold and the advantage of simplified decodability will be lost implying that the nonlinear lattice decoders [9] must be applied. On the other hand, since eq. (38) is verified independently from the orthogonality of the constellation, then the proposed scheme achieves full transmit diversity with non-orthogonal constellations as well.

III. Three transmit antennas with feedback

A. One bit feedback

1) Transmission Strategy: In this case, the signals transmitted from the first and second antennas keep the same expressions as in eq. (5) and eq. (6) respectively. The signal transmitted from the third antenna is given by:

$$s_3(t) = \sum_{m=1}^{M} \big[a_{1,\sigma(m)} w(t-(m-1)\delta) + a_{2,\sigma(m)} w(t-T_s-(m-1)\delta)\big] \quad (43)$$

where the choice of the function $\sigma(.)$ depends on the specific channel realization. This function will be chosen according to the following rule:

$$\sigma(.) = \begin{cases} \mathbf{1}(.), & \sum_{q=1}^{Q}\sum_{l=1}^{L} h_{q,1,l}h_{q,3,l} \geq 0; \\ \pi(.), & \text{otherwise}. \end{cases} \quad (44)$$

where $\mathbf{1}(.)$ stands for the identity transformation and $\pi(.)$ is given in eq. (3).

Equations (43) and (44) show that the third antenna transmits either exactly the same signal as the first antenna or a permuted version of what was transmitted from this antenna. The reason behind the choice of the function σ given in eq. (44) is to couple the first and third antennas. In other words, σ is chosen in such a way that the signals transmitted from these antennas combine constructively. This choice will be further clarified in what follows. Note that the constructive interference will be achieved without inverting the polarities of the transmitted pulses.

2) Optimal ML Decoding: As in section II-B, assume that the second PPM symbol a_2 is in the n-th slot. For $\sigma(.) = \mathbf{1}(.)$, equations (12)-(15) can be written as:

$$\begin{cases} y_{q,l,1,m} = (h_{q,1,l}+h_{q,3,l})a_{1,m} + n_{q,l,1,m} \\ y_{q,l,2,m} = h_{q,2,l}a_{1,m} + n_{q,l,2,m} \end{cases} \quad (45)$$

for $m \notin \{2n-1, 2n\}$ and

$$\begin{cases} (y_{q,l,1,m} - h_{q,2,l}) = (h_{q,1,l}+h_{q,3,l})a_{1,m} - h_{q,2,l}a_{2,m} + n_{q,l,1,m} \\ y_{q,l,2,m} = h_{q,2,l}a_{1,m} + (h_{q,1,l}+h_{q,3,l})a_{2,m} + n_{q,l,2,m} \end{cases} \quad (46)$$

for $m \in \{2n-1, 2n\}$.

Consequently, the first information symbol can be determined from the following M decision variables:

$$b_{1,m} = \sum_{q,l} \left[(h_{q,1,l}+h_{q,3,l})y_{q,l,1,m} + h_{q,2,l}y_{q,l,2,m}\right] \quad (47)$$

for $m \notin \{2n-1, 2n\}$ and

$$b_{1,m} = \sum_{q,l} \left[(h_{q,1,l}+h_{q,3,l})(y_{q,l,1,m} - h_{q,2,l})h_{q,2,l}y_{q,l,2,m}\right] \quad (48)$$

for $m \in \{2n-1, 2n\}$.

In the same way, the second information symbol can be determined from the following two decision variables:

$$b_{2,m} = \sum_{q=1}^{Q}\sum_{l=1}^{L} [-h_{q,2,l}(y_{q,l,1,m} - h_{q,2,l}) + (h_{q,1,l}+h_{q,3,l})y_{q,l,2,m}] \quad ; \quad m \in \{2n-1, 2n\} \quad (49)$$

Note that in the absence of noise, equations (47), (48) and (49) imply that (for $i = 1, 2$):

$$b_{i,m} = \begin{cases} \sum_{q,l}(h_{q,1,l}^2 + h_{q,2,l}^2 + h_{q,3,l}^2) + 2\sum_{q,l} h_{q,1,l}h_{q,3,l}, & m = p_i; \\ 0, & m \neq p_i. \end{cases} \quad (50)$$

Note that $\sigma(.)$ is chosen to be equal to $\mathbf{1}(.)$ when $\sum_{q,l} h_{q,1,l}h_{q,3,l} \geq 0$ implying that $b_{i,p_i} \geq \sum_{q,l}\left(h_{q,1,l}^2 + h_{q,2,l}^2 + h_{q,3,l}^2\right)$ and showing that the destructive interference between the three transmit antennas is removed and that the SNR is maximized.

In the presence of a 1-bit feedback and for $\sigma(.) = \mathbf{1}(.)$, the decoding procedures described in eq. (22), (23) and (26) will remain unchanged. However, now, the decision variables $\{b_{1,m}^{(0)}, b_{2,m}^{(0)}, b_{2,m}^{(1)}\}_{m=1}^{M}$ and the constant K_1 given in eq. (18) and (20) will take the following values: $b_{1,m}^{(0)} = \sum_{q,l}\left[(h_{q,1,l}+h_{q,3,l})y_{q,l,1,m} + h_{q,2,l}y_{q,l,2,m}\right]$, $b_{2,m}^{(0)} = \sum_{q,l}(h_{q,1,l}+h_{q,3,l})y_{q,l,2,m}$, $b_{2,m}^{(1)} = \sum_{q,l} h_{q,2,l}y_{q,l,1,m}$ and $K_1 = -\sum_{q,l}(h_{q,1,l}+h_{q,3,l})h_{q,2,l}$. Note that these relations follow from comparing eq. (18) to eq. (47,48) and eq. (20) to eq. (49).

Always assuming that the second symbol is in the n-th slot, when $\sum_{q,l} h_{q,1,l}h_{q,3,l} < 0$, $\sigma(.)$ is chosen to be equal to $\pi(.)$ and equations (12)-(15) will be written as:

$$\begin{cases} y_{q,l,1,m} - h_{q,3,l} = (h_{q,1,l}-h_{q,3,l})a_{1,m} + n_{q,l,1,m} \\ y_{q,l,2,m} = h_{q,2,l}a_{1,m} + n_{q,l,2,m} \end{cases} \quad (51)$$

for $m \notin \{2n-1, 2n\}$ and

$$\begin{cases} y_{q,l,1,m} - h_{q,2,l} - h_{q,3,l} = (h_{q,1,l}-h_{q,3,l})a_{1,m} - h_{q,2,l}a_{2,m} + n_{q,l,1,m} \\ y_{q,l,2,m} - h_{q,3,l} = h_{q,2,l}a_{1,m} + (h_{q,1,l}-h_{q,3,l})a_{2,m} + n_{q,l,2,m} \end{cases} \quad (52)$$

for $m \in \{2n-1, 2n\}$.

The above equations show that the information symbols can be determined from the decision variables shown in equations (53) and (54) at the bottom of the page.

This implies that eq. (22), (23) and (26) can be applied with: $b_{1,m}^{(0)} = \sum_{q,l}\left[(h_{q,1,l}-h_{q,3,l})(y_{q,l,1,m}-h_{q,3,l}) + h_{q,2,l}y_{q,l,2,m}\right]$, $b_{2,m}^{(0)} = \sum_{q,l}(h_{q,1,l}-h_{q,3,l})y_{q,l,2,m}$, $b_{2,m}^{(1)} = \sum_{q,l} h_{q,2,l}y_{q,l,1,m}$ and $K_1 = \sum_{q,l}[-(h_{q,1,l}-h_{q,3,l})h_{q,2,l} - h_{q,2,l}h_{q,3,l}] = -\sum_{q,l} h_{q,1,l}h_{q,2,l}$.

$$b_{1,m} = \begin{cases} \sum_{q,l}\left[(h_{q,1,l}-h_{q,3,l})(y_{q,l,1,m}-h_{q,3,l}) + h_{q,2,l}y_{q,l,2,m}\right], & m \notin \{2n-1, 2n\}; \\ \sum_{q,l}\left[(h_{q,1,l}-h_{q,3,l})(y_{q,l,1,m}-h_{q,2,l}-h_{q,3,l}) + h_{q,2,l}(y_{q,l,2,m}-h_{q,3,l})\right], & m \in \{2n-1, 2n\}. \end{cases} \quad (53)$$

$$b_{2,m} = \sum_{q,l}\left[-h_{q,2,l}(y_{q,l,1,m}-h_{q,2,l}-h_{q,3,l}) + (h_{q,1,l}-h_{q,3,l})(y_{q,l,2,m}-h_{q,3,l})\right] \quad ; \quad m \in \{2n-1, 2n\} \quad (54)$$

$$b_{1,m} = \begin{cases} \sum_{q,l} \left[h_{q,1,l} y_{q,l,1,m} + (h_{q,2,l} + h_{q,3,l}) y_{q,l,2,m} \right], & m \notin \{2n-1, 2n\}; \\ \sum_{q,l} \left[h_{q,1,l} (y_{q,l,1,m} - h_{q,2,l} - h_{q,3,l}) + (h_{q,2,l} + h_{q,3,l}) y_{q,l,2,m} \right], & m \in \{2n-1, 2n\}. \end{cases} \quad (64)$$

$$b_{2,m} = \sum_{q,l} \left[-(h_{q,2,l} + h_{q,3,l})(y_{q,l,1,m} - h_{q,2,l} - h_{q,3,l}) + h_{q,1,l} y_{q,l,2,m} \right] \quad ; \quad m \in \{2n-1, 2n\} \quad (65)$$

Note that in the absence of noise, eq. (53) and eq. (54) imply that (for $i = 1, 2$):

$$b_{i,m} = \begin{cases} \sum_{q,l} \left(h_{q,1,l}^2 + h_{q,2,l}^2 + h_{q,3,l}^2 \right) - 2 \sum_{q,l} h_{q,1,l} h_{q,3,l}, & m = p_i; \\ 0, & m \neq p_i. \end{cases} \quad (55)$$

and, consequently, $b_{i,p_i} \geq \sum_{q,l} \left(h_{q,1,l}^2 + h_{q,2,l}^2 + h_{q,3,l}^2 \right)$ since $\sum_{q,l} h_{q,1,l} h_{q,3,l} < 0$. This justifies the choice $\sigma(.) = \pi(.)$ when the last quantity is negative.

3) Suboptimal Decoding: In this case, the decision variables given in eq. (9) and eq. (10) will take the following values:

$$y_{q,l,1,m} = \left[h_{q,1,l} a_{1,m} + h_{q,3,l} a_{1,\sigma(m)} \right] + h_{q,2,l} a_{2,\pi(m)} + n_{q,l,1,m} \quad (56)$$

$$y_{q,l,2,m} = h_{q,2,l} a_{1,m} + \left[h_{q,1,l} a_{2,m} + h_{q,3,l} a_{2,\sigma(m)} \right] + n_{q,l,2,m} \quad (57)$$

On the other hand, for the value of $s_{i,n}$ given in eq. (31) (and for $i = 1, 2$):

$$a_{i,\sigma(2n-1)} - a_{i,\sigma(2n)} = \begin{cases} s_{i,n}, & \sigma(.) = \mathbf{1}(.); \\ -s_{i,n}, & \sigma(.) = \pi(.). \end{cases} \quad (58)$$

Consequently, following from equations (56)-(58), the modified decision variables given in eq. (29) and eq. (30) will take the following values:

$$z_{q,l,1,n} = (h_{q,1,l} + c h_{q,3,l}) s_{1,n} - h_{q,2,l} s_{2,n} + n'_{q,l,1,n} \quad (59)$$

$$z_{q,l,2,n} = h_{q,2,l} s_{1,n} + (h_{q,1,l} + c h_{q,3,l}) s_{2,n} + n'_{q,l,2,n} \quad (60)$$

where:

$$c = \begin{cases} +1, & \sigma(.) = \mathbf{1}(.); \\ -1, & \sigma(.) = \pi(.). \end{cases} \quad (61)$$

Finally, replacing $h_{q,1,l}$ by $h_{q,1,l} + c h_{q,3,l}$ in eq. (34), we conclude that the final decision variables $Z_{i,n}$ are related to the information symbols by the following relation:

$$Z_{i,n} = \left[\sum_{q=1}^{Q} \sum_{p=1}^{3} \sum_{l=1}^{L} h_{q,p,l}^2 + 2c \sum_{q=1}^{Q} \sum_{l=1}^{L} h_{q,1,l} h_{q,3,l} \right] s_{i,n} + N_{i,n} \quad (62)$$

As indicated before, to maximize the SNR, a convenient choice of the function $\sigma(.)$ given in eq. (43) based on the feedback bit is: $\sigma(.) = \mathbf{1}(.)$ if $\sum_{q,l} h_{q,1,l} h_{q,3,l} \geq 0$ and $\sigma(.) = \pi(.)$ otherwise.

B. Two bits feedback

1) Transmission Strategy: In this case, the signals transmitted from the first two antennas are given in eq. (5) and eq. (6) respectively. When two bits of feedback are available, the transmitter chooses to couple either the first and third antennas or the second and third antennas depending on whether $|\sum_{q,l} h_{q,1,l} h_{q,3,l}|$ is greater than or less than $|\sum_{q,l} h_{q,2,l} h_{q,3,l}|$. In other words, if $|\sum_{q,l} h_{q,1,l} h_{q,3,l}| \geq |\sum_{q,l} h_{q,2,l} h_{q,3,l}|$, the third antenna transmits the signal given in eq. (43) and selects the function $\sigma(.)$ according to eq.(44). Otherwise, when $|\sum_{q,l} h_{q,1,l} h_{q,3,l}| < |\sum_{q,l} h_{q,2,l} h_{q,3,l}|$, the third antenna transmits the following signal:

$$s_3(t) = \sum_{m=1}^{M} \left[a_{2,\sigma(\pi(m))} w(t - (m-1)\delta) + a_{1,\sigma(m)} w(t - T_s - (m-1)\delta) \right] \quad (63)$$

where $\sigma(.)$ is chosen to be equal to $\mathbf{1}(.)$ (resp. $\pi(.)$) when $\sum_{q,l} h_{q,2,l} h_{q,3,l}$ is positive (resp. negative).

2) Optimal ML Decoding: When $|\sum_{q,l} h_{q,2,l} h_{q,3,l}| \geq |\sum_{q,l} h_{q,1,l} h_{q,3,l}|$ and $\sum_{q,l} h_{q,2,l} h_{q,3,l} > 0$, it can be proven that the information symbols can be determined from the decision variables given in equations (64) and (65) at the top of the page.

In this case, eq. (22), (23) and (26) can be applied with: $b_{1,m}^{(0)} = \sum_{q,l} [h_{q,1,l} y_{q,l,1,m} + (h_{q,2,l} + h_{q,3,l}) y_{q,l,2,m}]$, $b_{2,m}^{(0)} = \sum_{q,l} h_{q,1,l} y_{q,l,2,m}$, $b_{2,m}^{(1)} = \sum_{q,l} (h_{q,2,l} + h_{q,3,l}) y_{q,l,1,m}$ and $K_1 = -\sum_{q,l} h_{q,1,l} (h_{q,2,l} + h_{q,3,l})$.

On the other hand, when $\sigma(.) = \pi(.)$, the information symbols can be determined from the decision variables given in equations (66) and (67) at the bottom of the page. This implies that eq. (22), (23) and (26) can be applied with: $b_{1,m}^{(0)} = \sum_{q,l} [h_{q,1,l} y_{q,l,1,m} + (h_{q,2,l} - h_{q,3,l})(y_{q,l,2,m} - h_{q,3,l})]$, $b_{2,m}^{(0)} = \sum_{q,l} h_{q,1,l} y_{q,l,2,m}$, $b_{2,m}^{(1)} = \sum_{q,l} (h_{q,2,l} - h_{q,3,l}) y_{q,l,1,m}$ and $K_1 = -\sum_{q,l} h_{q,1,l} h_{q,2,l}$.

In both cases, it can be proven that (for $i = 1, 2$):

$$b_{i,m} = \begin{cases} \sum_{q,l} \left(h_{q,1,l}^2 + h_{q,2,l}^2 + h_{q,3,l}^2 \right) + 2c \sum_{q,l} h_{q,2,l} h_{q,3,l}, & m = p_i; \\ 0, & m \neq p_i. \end{cases} \quad (68)$$

$$b_{1,m} = \begin{cases} \sum_{q,l} \left[h_{q,1,l} y_{q,l,1,m} + (h_{q,2,l} - h_{q,3,l})(y_{q,l,2,m} - h_{q,3,l}) \right], & m \notin \{2n-1, 2n\}; \\ \sum_{q,l} \left[h_{q,1,l} (y_{q,l,1,m} - h_{q,2,l}) + (h_{q,2,l} - h_{q,3,l})(y_{q,l,2,m} - h_{q,3,l}) \right], & m \in \{2n-1, 2n\}. \end{cases} \quad (66)$$

$$b_{2,m} = \sum_{q,l} \left[-(h_{q,2,l} - h_{q,3,l})(y_{q,l,1,m} - h_{q,2,l}) + h_{q,1,l} (y_{q,l,2,m} - h_{q,3,l}) \right] \quad ; \quad m \in \{2n-1, 2n\} \quad (67)$$

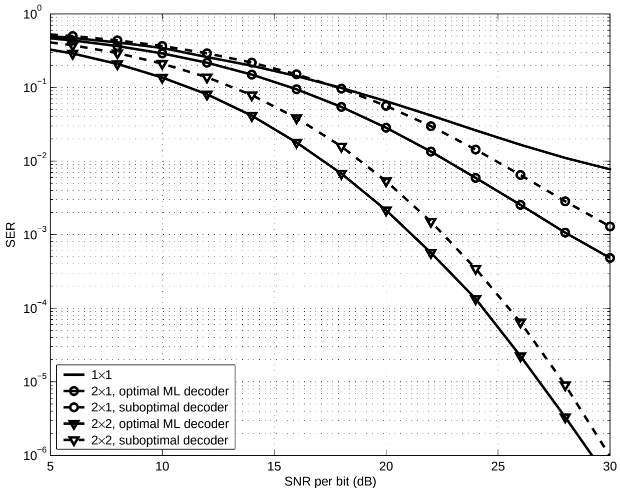

Fig. 1. The proposed scheme versus single-antenna systems with 4-PPM and a 1-finger Rake.

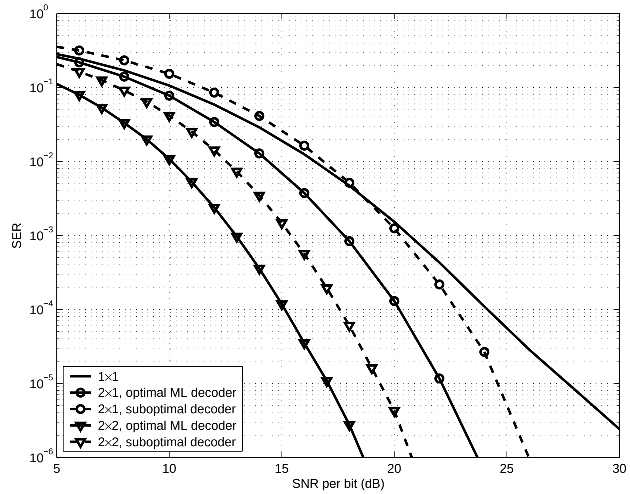

Fig. 2. The proposed scheme versus single-antenna systems with 4-PPM and a 20-finger Rake.

where $c = 1$ (resp. -1) when $\sum_{q,l} h_{q,2,l} h_{q,3,l}$ is positive (resp. negative). This shows that a full transmit diversity order of three is achieved.

3) Suboptimal Decoding: When the signal transmitted from the third antenna takes the value given in eq. (63), it can be easily proven that the decision variables associated with the suboptimal decoding procedure can be written as:

$$Z_{i,n} = \left[\sum_{q=1}^{Q}\sum_{p=1}^{3}\sum_{l=1}^{L} h_{q,p,l}^2 + 2c\sum_{q=1}^{Q}\sum_{l=1}^{L} h_{q,2,l} h_{q,3,l}\right] s_{i,n} + N_{i,n} \tag{69}$$

where c is defined in eq. (61). Equation (69) shows that the SNR is maximized and that the diversity order is enhanced with two bits of feedback.

C. Three bits feedback

In this case, the transmitter has the choice of transmitting the signals given in either eq. (6) and eq. (43), eq. (6) and eq. (63) or eq. (43) and eq. (6) from the second and third antennas respectively. In the last case, the first antenna is coupled with the second antenna. The signal transmitted from the first antenna always takes the value given in eq. (5).

The selection among these three possibilities depends on the value of $(i, j) \in \{(1, 3), (2, 3), (1, 2)\}$ that maximizes $|\sum_{q,l} h_{q,i,l} h_{q,j,l}|$. The mapping function is chosen as: $\sigma \equiv \mathbf{1}$ when $\sum_{q,l} h_{q,\tilde{i},l} h_{q,\tilde{j},l} \geq 0$ and $\sigma \equiv \pi$ otherwise where $(\tilde{i}, \tilde{j})$ is the value of (i, j) that maximizes $|\sum_{q,l} h_{q,i,l} h_{q,j,l}|$. Whether with optimal or suboptimal decoding, the corresponding decision variables are similar to those obtained with one bit and two bits of feedback and are omitted here for brevity.

IV. Simulations and results

Simulations are performed over the IEEE 802.15.3a channel model recommendation CM2 [13]. To insure the orthogonality of the received constellation, the modulation delay is chosen as $\delta = 100$ ns which is larger than the maximum delay spread of the UWB channel [13] (readers are referred to [8] for more details on the simulation setup).

Figures 1 and 2 show the performance of the proposed 2×2 ST code with 1-finger and 20-finger Rakes respectively. In these figures, we compare the performance of the optimal and suboptimal decoders with 4-PPM. Results show the high performance levels and the enhanced diversity order achieved by the proposed scheme. Results also show that the suboptimal decoder preserves diversity since the error curves corresponding to the optimal and suboptimal decoders are parallel to each other at high SNR. Moreover, at low SNRs, single-antenna systems might slightly outperform 2×1 systems associated with the suboptimal decoder. However, at high SNRs, the latter system will always achieve lower error rates.

Fig. 3 shows the performance of 8-PPM with receivers that are equipped with a 10-finger Rake and suboptimal decoders. As expected, increasing the number of feedback bits improves the error performance of the 3×1 systems. The highest improvement results from the first feedback bit. Compared to this improvement, the additional feedback bits result in marginal gains. Similar results are observed in Fig. 4 when applying the optimal decoder.

In Fig. 5 we compare the complexity of the proposed decoders with respect to the PPM-specific lattice decoder proposed in [9]. Note that this decoder is the most popular decoder used to decode all the existing unipolar PPM ST codes [8]. In this figure we plot the average time needed for decoding one information symbol as a function the signal-set dimensionality (which is equal to the number of PPM positions). In this simulation setup, the proposed 2×2 code is applied and the receiver is equipped with a 5-finger Rake. Note that unlike the lattice decoders that necessitate long convergence times at low SNRs, the decoding times of the proposed optimal and suboptimal decoders are the same for all values of the SNR. The superiority of the proposed decoders in terms of complexity is evident. Finally, note that the gap between the proposed solutions and [9] increases with the dimensionality of the PPM constellation and with the SNR.

To highlight the advantages of ST coding with UWB, Fig. 6 compares systems having the same overall diversity order

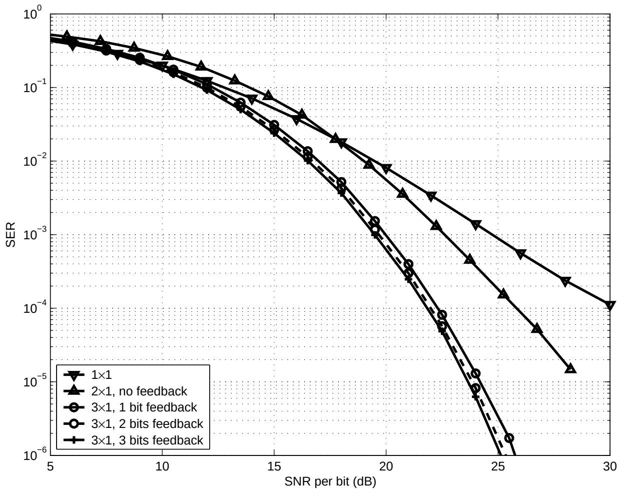

Fig. 3. Performance of 8-PPM with 1 receive antenna and a 10-finger Rake. The suboptimal decoder is applied with the 2×1 and 3×1 systems.

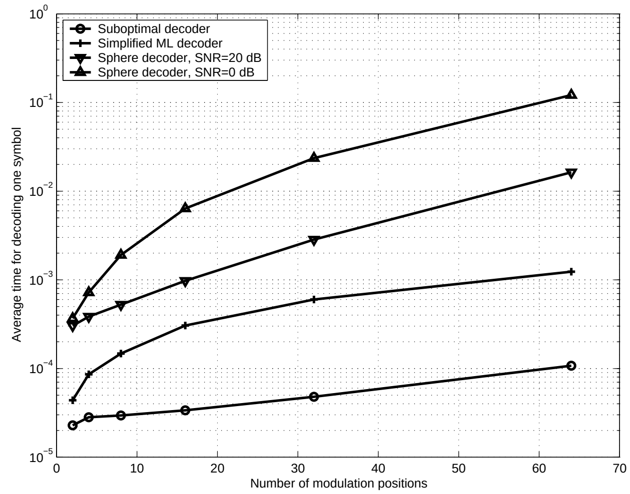

Fig. 5. Complexity of the proposed decoders compared to that of the decoder in [9] with 2×1 systems. The receiver is equipped with a 5-finger Rake.

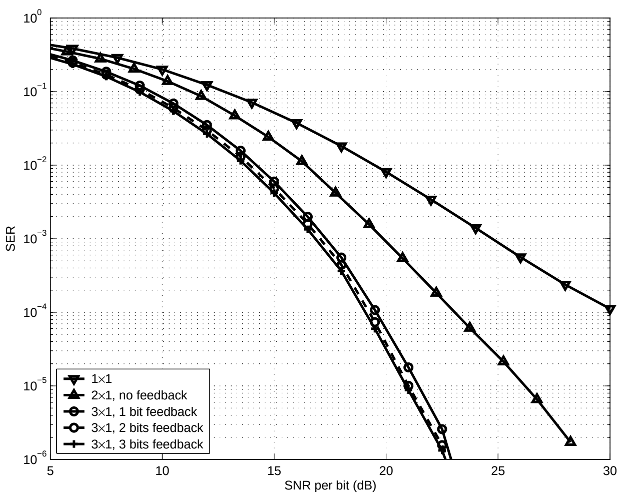

Fig. 4. Performance of 8-PPM with 1 receive antenna and a 10-finger Rake. The optimal decoder is applied with the 2×1 and 3×1 systems.

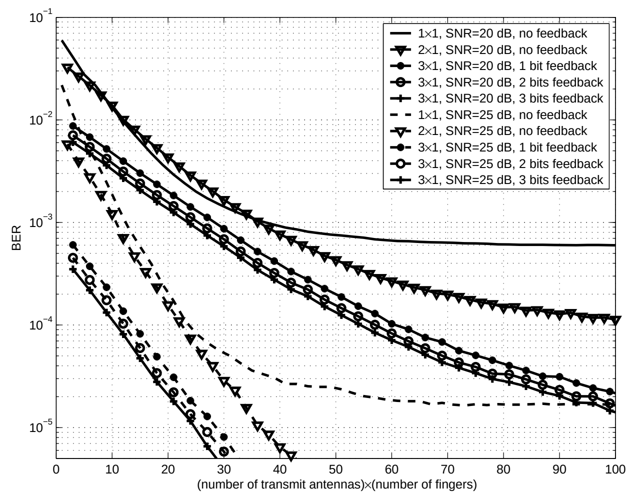

Fig. 6. Transmit diversity versus multi-path diversity with 2-PPM. The suboptimal decoder is applied with the 2×1 and 3×1 systems.

that is equal to PQL (P is the number of transmit antennas). 2-PPM is used and the suboptimal decoder is applied with the 2×1 and the 3×1 systems. For a fair comparison, we plot the bit error rates (BER) as a function of PL. For example, a 1×1 system with 60 fingers achieves a BER of 7×10^{-4} at 20 dB. In this case, the 2×1 system with only 30 fingers achieves a better BER in the order of 3×10^{-4}. Fig. 6 shows that exploiting the transmit diversity by increasing the number of transmit antennas can be more beneficial than enhancing the multi-path diversity by increasing the number of Rake fingers even though there is no increase in the energy capture. This follows from the fact that consecutive multi-path components of the same sub-channel can be simultaneously faded because of cluster and channel shadowing [13].

V. Conclusion

We investigated the problem of ST coding with TH-UWB systems using PPM. The proposed scheme has a full rate and is fully diverse resulting in high performance levels over the realistic indoor UWB channels. Moreover, this scheme is adapted to unipolar transmissions and, consequently, does not necessitate additional constraints on the RF circuitry to control the phase or the amplitude of the very low duty cycle sub-nanosecond pulses. At the receiver, a simple ML decoder whose complexity grows linearly with the signal-set dimensionality assures a fast and optimal separation of the transmitted data streams. The shape preserving constraint renders the proposed code applicable with optical wireless communications as well.

Appendix

The summation on the right hand side of eq. (24) can be written as:

$$\mathcal{S}(n) \triangleq \sum_{j=1}^{8} \mathcal{S}_j(n) \tag{70}$$

where:

$$\mathcal{S}_1(n) \triangleq \sum_{q,l,m} \left[y_{q,l,1,m}^2 + y_{q,l,2,m}^2 \right] \tag{71}$$

$$\mathcal{S}_2(n) \triangleq \sum_{q,l} \left(h_{q,1,l}^2 + h_{q,2,l}^2\right) \sum_m (\tilde{a}_{1,m}(n))^2 = \sum_{q,l} \left(h_{q,1,l}^2 + h_{q,2,l}^2\right) \tag{72}$$

$$\mathcal{S}_3(n) \triangleq \sum_{q,l} h_{q,1,l}^2 \sum_m (\tilde{a}_{2,m}(n))^2 = \sum_{q,l} h_{q,1,l}^2 \tag{73}$$

$$\mathcal{S}_4(n) \triangleq \sum_{q,l} h_{q,2,l}^2 \sum_m (\tilde{a}_{2,\pi(m)}(n))^2 \tag{74}$$

$$= \sum_{q,l} h_{q,2,l}^2 \sum_m (\tilde{a}_{2,m}(n))^2 = \sum_{q,l} h_{q,2,l}^2 \tag{75}$$

$$\mathcal{S}_5(n) \triangleq -2 \sum_{q,l,m} \left(h_{q,1,l} y_{q,l,1,m} + h_{q,2,l} y_{q,l,2,m}\right) \tilde{a}_{1,m}(n) \tag{76}$$

$$= -2 \sum_{q,l} \left(h_{q,1,l} y_{q,l,1,\tilde{p}_1(n)} + h_{q,2,l} y_{q,l,2,\tilde{p}_1(n)}\right) \tag{77}$$

$$\mathcal{S}_6(n) \triangleq -2 \sum_{q,l,m} h_{q,1,l} y_{q,l,2,m} \tilde{a}_{2,m}(n) \tag{78}$$

$$= -2 \sum_{q,l} h_{q,1,l} y_{q,l,2,\tilde{p}_2(n)} \tag{79}$$

$$\mathcal{S}_7(n) \triangleq -2 \sum_{q,l,m} h_{q,2,l} y_{q,l,1,m} \tilde{a}_{2,\pi(m)}(n) \tag{80}$$

$$= -2 \sum_{q,l,m} h_{q,2,l} y_{q,l,1,\pi^{-1}(m)} \tilde{a}_{2,m}(n) \tag{81}$$

$$= -2 \sum_{q,l,m} h_{q,2,l} y_{q,l,1,\pi(m)} \tilde{a}_{2,m}(n) \tag{82}$$

$$= -2 \sum_{q,l} h_{q,2,l} y_{q,l,1,\pi(\tilde{p}_2(n))} \tag{83}$$

$$\mathcal{S}_8(n) \triangleq 2 \sum_{q,l} h_{q,1,l} h_{q,2,l} \sum_m \tilde{a}_{1,m}(n) \left(\tilde{a}_{2,m}(n) + \tilde{a}_{2,\pi(m)}(n)\right) \tag{84}$$

$$= 2 \sum_{q,l} h_{q,1,l} h_{q,2,l} \left(\tilde{a}_{2,\tilde{p}_1(n)}(n) + \tilde{a}_{2,\pi(\tilde{p}_1(n))}(n)\right) \tag{85}$$

$$= 2\delta_{\lceil \frac{\tilde{p}_1(n)}{2} \rceil, n} \sum_{q,l} h_{q,1,l} h_{q,2,l} \tag{86}$$

where $\delta_{i,j}$ stands for Kronecker's delta function ($\delta_{i,j} = 1$ for $i = j$ and $\delta_{i,j} = 0$ for $i \neq j$).

Note that the second equalities in eq. (72) and eq. (73) follow from the fact that only one component of the vector $\tilde{a}_i(n) = [\tilde{a}_{i,1}(n) \ \cdots \ \tilde{a}_{i,M}(n)]^T$ is equal to 1 while the remaining components are equal to 0 for $i = 1, 2$. The first equality in eq. (75) follows from replacing m by $\pi^{-1}(m)$ and by observing that $\{\pi^{-1}(m) \ ; \ m = 1, \dots, M\}$ is equal to the set $\{1, \dots, M\}$. Equations (77) and (79) follow from the fact that, by construction, $\tilde{a}_{i,m}(n)$ is different from zero (and equal to 1) only for $m = \tilde{p}_i(n)$.

Equation (81) follows from performing the change of variable $m \rightarrow \pi^{-1}(m)$. Equation (82) follows from the fact that $\pi^{-1}(m) = \pi(m)$ for all values of $m \in \{1, \dots, M\}$. Finally, eq. (86) follows from the fact that the term $\left(\tilde{a}_{2,\tilde{p}_1(n)}(n) + \tilde{a}_{2,\pi(\tilde{p}_1(n))}(n)\right)$ is different from zero (and equal to 1) only if $\tilde{p}_1(n)$ corresponds to a position in the n-th slot. In other words, $\left(\tilde{a}_{2,\tilde{p}_1(n)}(n) + \tilde{a}_{2,\pi(\tilde{p}_1(n))}(n)\right) = 1$ if and only if $\tilde{p}_1(n) \in \{2n-1, 2n\}$ implying that $\lceil \tilde{p}_1(n)/2 \rceil = n$ where the function $\lceil x \rceil$ rounds the real number x to the nearest integer that is greater than or equal to it.

Now, equations (71)-(75) show that the summations $\mathcal{S}_1(n), \dots, \mathcal{S}_4(n)$ are independent from the information symbols and hence can be omitted from the decision metric in eq. (24). Consequently, the optimal ML decoder decides in favor of the slot index n that minimizes the summation $\sum_{j=5}^{8} \mathcal{S}_j(n)$. Equations (77)-(86) show that this is equivalent to decide in favor of the value of n that maximizes:

$$\sum_{q=1}^{Q} \sum_{l=1}^{L} \Big[h_{q,1,l} \left(y_{q,l,1,\tilde{p}_1(n)} + y_{q,l,2,\tilde{p}_2(n)}\right) + h_{q,2,l} \left(y_{q,l,2,\tilde{p}_1(n)} + y_{q,l,1,\pi(\tilde{p}_2(n))}\right) - h_{q,1,l} h_{q,2,l} \delta_{\lceil \tilde{p}_1(n)/2 \rceil, n} \Big]$$

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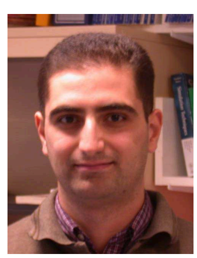

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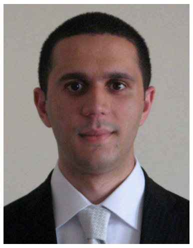

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