On Space-Time Coding with Pulse Position and Amplitude Modulations for Time-Hopping Ultra-Wideband Systems

Chadi Abou-Rjeily, *Member IEEE* and Jean-Claude Belfiore, *Member IEEE*

***Abstract*— In this work, we propose novel families of space-time (ST) block codes that can be associated with impulse radio ultra-wideband (IR-UWB) communication systems. The carrier-less nature of this nonconventional totally-real transmission technique necessitates the construction of new suitable coding schemes. In fact, the last generation of complex-valued ST codes (namely the perfect codes) can not be associated with IR-UWB systems where the phase reconstitution at the receiver side is practically infeasible. On the other hand, while the perfect codes were considered mainly with QAM and HEX constellations, IR-UWB systems are often associated with PPM and hybrid PPM-PAM constellations. In this paper, instead of adopting the classical approach of constructing ST codes over infinite fields ($\mathbb{Z}[i]$ or $\mathbb{Z}[j]$ for the perfect codes), we study the possibility of constructing modulation-specific codes that are exclusive to PPM and PPM-PAM. The proposed full-rate codes are totally-real, information-lossless and have a uniform average energy per transmit antenna. They permit to achieve a full diversity order with any number of transmit antennas. In some situations, the proposed schemes have an optimal non-vanishing coding gain and satisfy all the construction constraints of the perfect codes in addition to the constraint of being totally-real. Simulations performed over realistic indoor UWB channels showed that the proposed schemes outperform the best-known codes constructed from cyclic division algebras.**

***Index Terms*— UWB, Space-Time, MIMO, PPM, PAM, PPAM, modulation-specific codes.**

I. Introduction and problem formulation

The literature of space-time (ST) coding is huge. However, associating ST coding with impulse radio (IR) UWB is a recent and challenging research area [2]–[5]. This approach can be a candidate solution for very high data rate wireless personal area networks (WPANs). However, an additional constraint related to the nature of the totally-real carrier-less transmissions must be added. Because of this constraint, the best known ST codes (namely the golden code [6] and the perfect codes [1], [7]) that can be associated with all narrow-band and wide-band CDMA and OFDM systems can not be applied with this nonconventional transmission technique. While phase rotations can be exploited in order to achieve the transmit diversity in conventional narrow-band and wide-band systems, the very large bandwidth occupied by the transmitted UWB pulses renders the phase reconstitution at the receiver side practically infeasible. This motivates the construction of new families of totally-real ST coding schemes.

Given the high multi-path diversity offered by the UWB channels, the utility of an additional spatial diversity may be questionable. The benefits of merging multi-antenna techniques with UWB systems over realistic indoor channels [8] was outlined in different contributions [3], [4], [9]. Despite the high frequency selectivity of the UWB channels, profiting from multi-path diversity necessitates Rake receivers with very high orders. This follows from the very important delay spread of these channels. In this context, the additional spatial degree of freedom can result in higher performance levels, multiplexing gains and communication ranges. Given the low admissible transmission levels imposed by the various normalization organizations all over the world [10], the last point can be very critical for UWB systems. In this context, multi-antenna UWB systems can be one solution to the growing demand of a wide variety of indoor wireless applications for high-rate reliable communications.

One possible solution for constructing full rate, fully diverse and totally-real ST codes is the application of the construction procedures that were adopted for constructing the last generation of ST codes [1], [6], [7]. In particular, cyclic division algebras (CDA) [11]–[15] are appealing because they result in a systematic code design. Moreover, the best known codes are based on CDA. Designate by $\mathbb{F}$ the infinite extension of the transmitted constellation (for example $\mathbb{F} = \mathbb{Q}$ for PAM and $\mathbb{F} = \mathbb{Q}(i)$ for QAM). The construction of $n \times n$ codewords calls for the choice of a cyclic field extension $\mathbb{K}/\mathbb{F}$ of degree n whose Galois group is given by $Gal(\mathbb{K}/\mathbb{F}) = \langle\sigma\rangle$ with $\sigma^n = 1$. Elements of the cyclic algebra $A(\mathbb{K}/\mathbb{F}, \sigma, \gamma)$ have the following matrix representation:

$$C(k_0, \dots, k_{n-1}) = \begin{bmatrix} k_0 & k_1 & \cdots & k_{n-1} \\ \gamma\sigma(k_{n-1}) & \sigma(k_0) & \cdots & \sigma(k_{n-2}) \\ \vdots & \ddots & \ddots & \vdots \\ \gamma\sigma^{n-1}(k_1) & \cdots & \gamma\sigma^{n-1}(k_{n-1}) & \sigma^{n-1}(k_0) \end{bmatrix} \quad (1)$$

where $k_0, \dots, k_{n-1} \in \mathbb{K}$. γ is chosen such that there is no element in $\mathbb{K}$ whose norm is equal to γ^t for $t = 1, \dots, n-1$. Let $\mathbb{K} = \mathbb{F}(\theta)$ and designate by $a_1, \dots, a_{n^2}$ the information symbols that belong to a constellation carved from $\mathbb{F}$, then:

$$k_i = \sum_{j=0}^{n-1} a_{ni+j+1}\theta^j \quad (2)$$

C. Abou-Rjeily is with the Faculty of Engineering and Architecture of the Lebanese American University (LAU), P.O.B. 36 Byblos, Lebanon. (e-mail: chadi.abourjeily@lau.edu.lb).

J.-C. Belfiore is with the COMELEC department, École Nationale Supérieure des Télécommunications (ENST), 46 rue Barrault 75013 Paris France. (e-mail: belfiore@enst.fr).

When $\gamma, a_1, \dots, a_{n^2} \in \mathcal{O}_{\mathbb{F}}$ (the ring of integers of $\mathbb{F}$), then the resulting code has a non-vanishing determinant [13], [15]. Designate by G the $n \times n$ lattice generator matrix whose (i, j)-th element is equal to $\sigma^{i-1}(\theta^{j-1})$. The perfect codes are based on CDAs and they verify a large number of interesting properties:

- They achieve full rate and full diversity.
- They have the best known non-vanishing coding gains.
- They do not introduce any shaping losses (G is unitary).
- They are information lossless.
- They are energy efficient codes that have a constant average transmit energy per antenna ($|\gamma| = 1$).

For example, the last property was verified by choosing $\gamma = i$ in [6] and $\gamma = \frac{a+ib}{b+ai}$ in [7] where $a + ib$ is a non-norm element in the considered field extension. This property results in important performance gains. In fact, when $|\gamma| = 1$, the different layers of the codeword given in eq. (1) are equally protected against the error events (even ST codes that do not have a non-vanishing determinant were based on transcendental numbers whose magnitudes are equal to 1 [12]).

For IR-UWB, γ can not be a complex number. In fact, when γ is complex, it corresponds to a phase rotation. On the other hand, the transmitted UWB pulses occupy a bandwidth that extends over several GHz. Consequently, applying a phase rotation (whose value is given by γ) on one frequency component will not result in the same phase rotation on another component (given that these components can be separated by several GHz). In other words, a phase rotation of γ can not be applied simultaneously on the different components of the spectrum without dividing this band into several smaller sub-bands and applying the rotation in each one of them. Since this approach contradicts the nature of the time domain IR-UWB transmissions, we conclude that IR-UWB-specific codes must be totally-real.

For totally-real codes that can be used in conjunction with IR-UWB, γ, $\mathbb{F}$ and $\mathbb{K}$ must be real. However, for totally-real constructions, energy efficiency can be obtained uniquely by choosing $\gamma = \pm 1$. This shows the non-existence of totally-real energy-efficient codes based on CDAs for $n \geq 3$ since $\gamma^2 = 1$ is always a norm. For $n = 2$, energy efficiency comes at the expense of shaping losses and vice versa (when $\gamma = -1$ is not a norm in $\mathbb{K}$, the generator matrix G can not be unitary and vice versa [5]).

Moreover, choosing $|\gamma| \neq 1$ results in energy-unbalanced codes that do not have a uniform average energy per transmit antenna resulting in performance losses. For example, $\gamma = 2$ results in a non-vanishing determinant when $\mathbb{K}$ is the maximal real subfield of the cyclotomic field with $n \in \{2, \dots, 6\}$ [5]. However, even though this choice maximizes the coding gain with $n \in \{2, 3\}$, the resulting coding gain is only a fraction of the coding gain achieved by the perfect codes [5].

On the other hand, IR-UWB is used in conjunction with pulse amplitude modulation (PAM) and (or) pulse position modulation (PPM). In what follows, we use hybrid M-PPM-M'-PAM constellations where the amplitude of the pulse transmitted during each position can take M' possible values. This modulation scheme is appealing since it takes advantage of the high temporal resolution to deliver higher data rates with lower complexity [16]. This constellation is given by:

$$\mathcal{C} = \{(2m' - 1 - M')e_{m+1} \ ; \ m' = 1, \dots, M' \ ; \ m = 0, \dots, M-1\} \quad (3)$$

where e_m is the m-th column of the $M \times M$ identity matrix I_M. $\mathcal{C}$ entails PPM and PAM as special cases.

Equation (1) can be readily applied with PPM-PAM. In this case, the scalars $a_1, \dots, a_{n^2}$ in eq. (2) are replaced by M-dimensional vectors taken from $\mathcal{C}$ in eq. (3). In this case, the codeword in eq. (1) becomes a $nM \times n$ matrix whose $((n' - 1)M + m, n'')$-th entry corresponds to the amplitude of the pulse (if any) transmitted at the m-th position of the n'-th antenna during the n''-th symbol duration for $m = 1, \dots, M$ and $n', n'' \in \{1, \dots, n\}$. Suppose that γ is an algebraic non-norm integer and designate by d_∞ and $d_{M,M'}$ the coding gain of eq. (1) over $\mathbb{Q}$ and over M-PPM-M'-PAM constellations respectively, then it was proven in [5] that:

$$d_{M,M'} = \begin{cases} 4d_\infty & ; \quad M = 1, \forall M' \\ 2d_\infty & ; \quad M > 1, \forall M' \end{cases} \quad (4)$$

In Fig. 1, we plot a 3-PPM-4-PAM constellation and a 3-dimensional extension of the 4-PAM (which, for example, corresponds to the constellation transmitted in the case of spatial multiplexing with 3 transmit antennas). This figure shows the degree of sparsity and the non-linearity of the PPM-PAM constellations. This particular structure can be further exploited in order to leverage the performance of the encoding scheme.

In this work, instead of adopting the classical approach of constructing ST codes over infinite fields ($\mathbb{Z}$ in the totally-real case), we profit from the particular structure of the PPM-PAM constellations in order to construct new coding schemes suitable for these modulations. The proposed schemes are energy-efficient, information-lossless and they introduce no shaping losses. However, unlike codes from CDA that can be associated with any PPM, PAM or hybrid PPM-PAM constellation, the new codes are exclusive to M-PPM-M'-PAM constellations for specific values of M. Finally, we show that for certain values of M, it is possible to construct ST codes that satisfy all the construction constraints of the perfect codes in addition to the constraint of being totally-real (which was impossible with codes from CDA for all constellations). A fresh look is provided on the construction of PPM-PAM-specific codes. The additional degree of freedom that results from the non-linearity of PPM-PAM constellations is exploited in order to overcome the constraint of having totally-real transmissions. The new modulation-specific codes show the best known performance among the existing Space-Time (ST) UWB schemes. It turns out that constructing modulation-specific codes might be more beneficial than the extension of modulation-nonspecific codes to PPM-PAM constellations.

Note that energy balanced schemes that are based on CDAs were proposed in [5]. However, unlike these schemes that take advantage of the pulse repetitions in Time-Hopping UWB (TH-UWB) in order to introduce inter-pulse coding or joint inter-symbol and inter-pulse coding and balance the transmitted codewords, the codes that we propose in this work

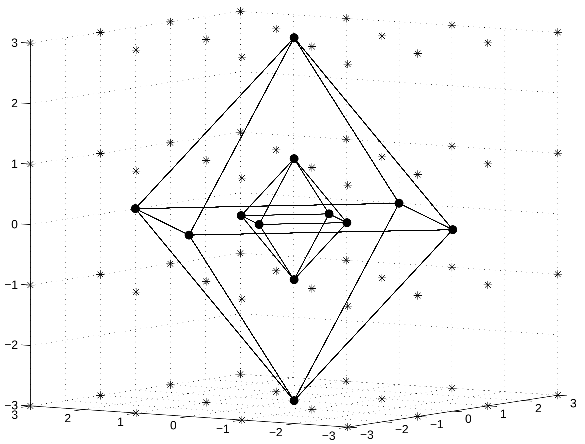

Fig. 1. The 3-PPM-4-PAM constellation (in filled circles). The stars corresponds to a $\{4-\text{PAM}\}^3$ constellation.

can be applied even in the absence of pulse repetitions (and consequently they can be used with very high data rate TH-UWB systems).

The rest of the paper is organized as follows. In section II, we present the system model of multi-antenna TH-UWB systems. The two families of modulation-specific codes are presented in section III and section IV. The proposed codes are comparisons with the ST codes from CDAs in section V. Simulations over realistic indoor UWB channels are represented in section VI while section VII concludes.

Notations: $I_{m\times n}$ denotes the first m columns of the $n\times n$ identity matrix I_n. $\mathbf{1}_{m\times n}$ and $\mathbf{0}_{m\times n}$ correspond to the $m\times n$ matrices whose elements are all equal to 1 and 0 respectively. $\mathbf{0}_m$ stands for the m-dimensional vector whose components are equal to zero. $*$ stands for convolution, $\otimes$ for the Kronecker product and $\delta(.)$ for the Dirac delta function. $\mathbb{Q}$ denotes the field of rational numbers. The functions $\text{N}_{\mathbb{K}/\mathbb{Q}}$ and $\text{Tr}_{\mathbb{K}/\mathbb{Q}}$ denote the norm and trace of an element in the field extension $\mathbb{K}/\mathbb{Q}$. $\text{diag}(X_1,\dots,X_n)$ corresponds to stacking the corresponding matrices on the principal diagonal. $\text{vec}(X)$ stands for stacking the columns of the matrix X vertically.

II. SYSTEM MODEL

Consider a single-user multi-antenna TH-UWB system with P transmit antennas and Q receive antennas. The antennas of the same user will share the same TH sequence, and since multiple access interference is not considered in this work, no reference to the TH sequence will be made hereafter. The signal transmitted from the p-th antenna can be expressed as:

$$s_p(t) = \frac{1}{\sqrt{PN_f}} a_p \sum_{n=0}^{N_f-1} w(t-nT_f-d_p\delta) \quad (5)$$

$$= \frac{1}{\sqrt{PN_f}} \sum_{n=0}^{N_f-1}\sum_{m=0}^{M-1} a_{p,m} w(t-nT_f-m\delta) \quad (6)$$

where a_p and d_p stand for the amplitude and the position of the symbol transmitted from the p-th transmit antenna. $a_{p,m} = a_p\delta(d_p-m)$ is the $(m+1)$-th component of the M-dimensional vector A_p that belongs to the set $\mathcal{C}$ given in eq. (3) for $p=1,\dots,P$ and $m=0,\dots,M-1$. In other words, $A_p=[a_{p,0},\dots,a_{p,M-1}]^T\in\mathcal{C}$ is composed of $M-1$ zero values and one component that belongs to the M'-ary PAM constellation. $w(t)$ is the monocycle pulse waveform of duration T_w normalized to have unit energy. N_f pulses are used to convey each information symbol. Each one of these pulses is emitted during one time frame of duration T_f. δ is the modulation delay and is chosen to verify $\delta\geq T_w$. In what follows, we suppose that T_f is larger than the channel delay spread and, consequently, the considered system does not suffer from inter frame interference (IFI) or inter symbol interference (ISI). The normalization factor P is introduced in order to insure that the total transmitted energy is the same as in the single-antenna case.

Note that eq. (6) models the TH-UWB systems that do not employ any kind of inter-pulse coding. In other words, the N_f pulses used to convey each information symbol have the same amplitudes and the same modulation positions. It also models very high data rate systems that do not employ any kind of pulse repetitions ($N_f=1$).

The signal received at the q-th antenna is given by:

$$r_q(t) = \sum_{p=1}^{P} s_p(t) * g_{q,p}(t-\varepsilon_{q,p}) + n_q(t) \quad (7)$$

$$= \frac{1}{\sqrt{PN_f}} \sum_{p=1}^{P}\sum_{n=0}^{N_f-1}\sum_{m=0}^{M-1} a_{p,m} h_{q,p}(t-nT_f-m\delta) + n_q(t) \quad (8)$$

where $n_q(t)$ is the noise at the q-th antenna which is supposed to be real AWGN with double sided spectral density $N_0/2$. $g_{q,p}(t)$ stands for the impulse response of the frequency selective channel between the p-th transmit antenna and the q-th receive antenna. $\varepsilon_{q,p}$ corresponds to the time delay of the first arriving multi-path component of $g_{q,p}(t)$. In fact, given the very small duration of the transmitted pulses, the propagation delays introduced by the spacing of the elements of the transmit or receive antenna arrays can be comparable with respect to T_w. These additional delays are calculated with respect to the first arriving ray. In other words, $\min_{p,q}(\varepsilon_{q,p})=0$. In what follows, we fix $h_{q,p}(t)=w(t)*g_{q,p}(t-\varepsilon_{q,p})$ implying that the receiver is synchronized to the first multi-path component between the transmit and the receive arrays.

In UWB systems, the interaction between the very short pulses is limited resulting in better immunity against multi-path fading. However, profiting from the multi-path diversity might necessitate the implementation of very high order Rake receivers given the very large delay spread of the UWB channels. In practice, TH-UWB systems are associated with receivers having a limited number of Rake fingers. Consider the case of a L-th order partial Rake (PRake) receiver that combines the L first arriving multi-path components [17]. Designate by Δ_l the delay of the l-th finger for $l=0,\dots,L-1$. At the receiver side, QLM decision variables are collected during each symbol duration. These decision variables are given by:

$$\begin{aligned} y_{q,l,m} &= \int_0^{N_f T_f} r_q(t)\widetilde{w}_{l,m}\mathrm{d}t \\ &= \sqrt{\frac{N_f}{P}} \sum_{p'=1}^{P} \sum_{m'=0}^{M-1} a_{p',m'} r_{q,p'}((m-m')\delta + \Delta_l) + n_{q,l,m} \end{aligned} \tag{9}$$

where the reference signal is given by:

$$\widetilde{w}_{l,m} = \sum_{n=0}^{N_f-1} w(t - nT_f - \Delta_l - m\delta) \tag{10}$$

Equation (9) follows from the condition of no IFI and no ISI and $r_{q,p}(\tau) = \int_0^{T_f} h_{q,p}(t)w(t-\tau)\mathrm{d}t$. The correlation between the zero-mean noise terms is given by:

$$E\left[n_{q,l,m}n_{q',l',m'}\right] = \frac{N_f N_0}{2}\gamma_w\left(\Delta_l - \Delta_{l'} + (m-m')\delta\right)\delta(q-q') \tag{11}$$

where $\gamma_w(\tau) = \int_0^{T_w} w(t)w(t-\tau)\mathrm{d}t$. From eq. (11), it follows that choosing $\Delta_l = lMT_w$ for $l = 0, \ldots, L-1$ results in white Gaussian noise terms since $\gamma_w(kT_w) = \delta(k)$. In what follows, the multiplying factor N_f will be removed from eq. (9) and eq. (11) since it has no impact on the signal to noise ratio. In the same way, the multiplying factor P can be removed from eq. (9) and included in the noise variance.

For a $P \times T$ space-time code, eq. (9) can be written in matrix form as:

$$Y = RC + N \tag{12}$$

where Y and N are $QLM \times T$ matrices corresponding to the decision variables and the noise terms respectively. C is the $PM \times T$ codeword whose $((p-1)M + m, t)$-th entry corresponds to the amplitude of the pulse (if any) transmitted at the m-th position of the p-th antenna during the t-th symbol duration for $p = 1, \ldots, P$, $m = 1, \ldots, M$ and $t = 1, \ldots, T$. $R = [R_1^T \cdots R_Q^T]^T$ is the $QLM \times PM$ channel matrix. $R_q = [R_{q,0}^T \cdots R_{q,L-1}^T]^T$ is a $LM \times PM$ matrix corresponding to the q-th receive antenna for $q = 1, \ldots, Q$. $R_{q,l} = [R_{q,l,1} \cdots R_{q,l,P}]$ is a $M \times PM$ for $l = 0, \ldots, L-1$. Finally, $R_{q,l,p}$ is a $M \times M$ matrix for $p = 1, \ldots, P$. The (m, m')-th element of $R_{q,p,l}$ corresponds to the impact of the signal transmitted during the m'-th position of the p-th antenna on the m-th correlator placed after the l-th Rake finger of the q-th receive antenna:

$$R_{q,p,l}(m, m') = r_{q,p}((m-m')\delta + \Delta_l) \tag{13}$$

Notice that increasing the spectral efficiency of the transmitted constellation by increasing the value of M results in high receiver complexity since the number of matched filters in eq. (10) is increased. However, this additional complexity is associated with better performance (measured by the achieved symbol error rate in this case). On the other hand, increasing the value of M' results in important performance losses without an additional increase in the number of matched filters deployed at the receiver.

III. Construction 1: Achieving full transmit diversity via position permutation

For M-PPM-M'-PAM with n transmit antennas, the minimal-delay codewords are given by:

$$C = \begin{bmatrix} k_0 & k_1 & k_2 & \cdots & k_{n-1} \\ \Omega\sigma(k_{n-1}) & \sigma(k_0) & \sigma(k_1) & \cdots & \sigma(k_{n-2}) \\ \Omega\sigma^2(k_{n-2}) & \Omega\sigma^2(k_{n-1}) & \sigma^2(k_0) & \cdots & \sigma^2(k_{n-3}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Omega\sigma^{n-1}(k_1) & \Omega\sigma^{n-1}(k_2) & \cdots & \Omega\sigma^{n-1}(k_{n-1}) & \sigma^{n-1}(k_0) \end{bmatrix} \tag{14}$$

where:

$$k_i = \sum_{j=0}^{n-1} a_{ni+j+1}\theta^j \in \mathcal{O}_{\mathbb{K}}^M \quad ; \quad a_1, \ldots, a_{n^2} \in \mathcal{C} \tag{15}$$

In comparison with eq. (1), the non-norm scalar γ is replaced by the $M \times M$ permutation matrix Ω given by:

$$\Omega = \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \tag{16}$$

It can be directly seen that the code in eq. (14) is not fully-diverse with PAM constellations ($M = 1$) since in this case eq. (14) corresponds to the code constructed from CDAs with $\Omega = \gamma = 1$ which is not a non-norm element.

A. Diversity Order

Proposition *3.1:* For n transmit antennas, the association of the codewords given in eq. (14) with the permutation matrix given in eq. (16) permits to achieve full transmit diversity with M-PPM-M'-PAM constellations for all values of M' and for $M \geq 2n-1$ or for the values of M verifying $\varphi(M) \geq n$ where $\varphi(.)$ stands for Euler's function.

For example, the proposed scheme achieves full diversity with M-PPM-M'-PAM constellations for all values of M' and for $M \geq 3$, $M \geq 5$, $\{M = 5, M \geq 7\}$, $\{M = 7, M \geq 9\}$ and $\{M = 7, 9, M \geq 11\}$ with $n = 2$, 3, 4, 5 and 6 transmit antennas respectively.

Proof: In order to get more insights on the properties of the proposed code, we next present the proof of Proposition 3.1 with $n = 2$. The proof for $n > 2$ is given in Appendix I.

Designate by $\Delta C(X_1, X_2)$ the difference between any two codewords given in eq. (14):

$$\Delta C(X_1, X_2) = C - C' = \begin{bmatrix} X_1 & X_2 \\ \Omega\sigma(X_2) & \sigma(X_1) \end{bmatrix} \tag{17}$$

where C and C' are the codewords associated with the vector pairs (k_0, k_1) and (k'_0, k'_1) respectively where k_0, k'_0, k_1, k'_1 are M-dimensional vectors whose structure verifies eq. (15). $X_1 = k_0 - k'_0$ and $X_2 = k_1 - k'_1$ belong to the set $\mathcal{A}$ given by:

$$\mathcal{A} = \{(a-a') + (b-b')\theta \mid a, a', b, b' \in \mathcal{C}\} \subset \mathcal{O}_{\mathbb{K}}^M \tag{18}$$

where $\mathcal{C}$ stands for the M-PPM-M'-PAM constellation given in eq. (3).

We will now show that all non-zero vector couples (X_1, X_2) that result in a rank deficient matrix $\Delta C(X_1, X_2)$ do not belong to the set $\mathcal{A}$ given in eq. (18) for all values of M'. In other words, according to the rank criterion given in [18], full transmit diversity is achieved with M-PPM-M'-PAM constellations if:

$$\text{rank}\left(\Delta C(X_1, X_2)\right) = 2 \ ; \ \forall \ (X_1, X_2) \in \mathcal{A}^2 \backslash \{(\mathbf{0}_M, \mathbf{0}_M)\} \tag{19}$$

where $\mathbf{0}_M$ is the M-dimensional all-zero vector. In what follows, $\Delta C(X_1, X_2)$ will be referred to as ΔC when there is no ambiguity. Denote by $X_{i,m}$ the m-th component of the vector X_i for $i = 1, 2$ and $m = 1, \ldots, M$.

Proposition: if $\exists \, i, m \,|\, X_{i,m} = 0$ then rank$(\Delta C(X_1, X_2)) = 2$ unless $X_1 = X_2 = \mathbf{0}_M$.

Proof: Designate by π the cyclic permutation given by:

$$\pi(i) = i \mod (M) + 1 \ ; \ i = 1, \ldots, M \tag{20}$$

π defines a bijection over $\{1, \ldots, M\}$ where $\pi^{-1}(i) = i - 2 \mod (M) + 1$ for $i = 1, \ldots, M$. The $2M \times 2$ matrix $\Delta C(X_1, X_2)$ can be written as:

$$\begin{bmatrix} X_{1,1} & \cdots & X_{1,M} & \sigma(X_{2,\pi^{-1}(1)}) & \cdots & \sigma(X_{2,\pi^{-1}(M)}) \\ X_{2,1} & \cdots & X_{2,M} & \sigma(X_{1,1}) & \cdots & \sigma(X_{1,M}) \end{bmatrix}^T \tag{21}$$

Suppose that $X_{1,m} = 0$ for a given value of m. When rank$(\Delta C) < 2$, the two columns of ΔC have the same direction. Therefore, considering the first M rows of ΔC, $X_{1,m} = 0 \Rightarrow X_{2,m} = 0$. Now we have $\sigma(X_{2,m}) = 0$ (since $X_{2,m} = 0$ and $\{1, \theta\}$ is an integral basis of $\mathcal{O}_{\mathbb{K}}$). Considering the last M rows of ΔC, $\sigma(X_{2,m}) = 0 \Rightarrow \sigma(X_{1,\pi(m)}) = 0 \Rightarrow X_{1,\pi(m)} = 0$. Starting the same procedure again with $\pi(m)$ rather than m, we conclude by iteration that $X_{1,m} = X_{1,\pi(m)} = \cdots = X_{1,\pi^{M-1}(m)} = 0$ and $X_{2,m} = X_{2,\pi(m)} = \cdots = X_{2,\pi^{M-1}(m)} = 0 \Leftrightarrow X_1 = X_2 = \mathbf{0}_M$ since π defines a bijection over $\{1, \ldots, M\}$. The same proof holds if $\exists m \,|\, X_{2,m} = 0$.

Lemma: For $n = 2$ transmit antennas, the code in eq. (14) achieves full diversity with M-PPM-M'-PAM constellations for $M > 4$ and for all values of M'.

Proof: From the definition of $\mathcal{A}$ in eq. (18), X_1 and X_2 are linear combinations of any four columns of the $M \times M$ identity matrix I_M. Therefore for $M > 4$, X_1 and X_2 each have at least one zero component resulting in full rank as shown in the last proposition.

We must now verify that ΔC has a full rank when all of its components are different from zero. In this case, $X_{1,m}, X_{2,m} \in \mathcal{O}^*_{\mathbb{K}} = \mathbb{Z}^* \oplus \theta\mathbb{Z}^* \oplus \mathcal{O}'_{\mathbb{K}}$ for all values of m where $\mathcal{O}'_{\mathbb{K}} = \{a + b\theta \mid a, b \in \mathbb{Z}^*\}$. When all the components of ΔC are different from zero, rank$(\Delta C) < 2$ implies that:

$$\frac{X_{2,1}}{X_{1,1}} = \cdots = \frac{X_{2,M}}{X_{1,M}} = \frac{\sigma(X_{1,1})}{\sigma(X_{2,\pi^{-1}(1)})} = \cdots = \frac{\sigma(X_{1,M})}{\sigma(X_{2,\pi^{-1}(M)})} = k \tag{22}$$

where k is any element of $\mathbb{K}$. After some manipulations, eq. (22) becomes:

$$X_{1,1} = \left(\text{N}_{\mathbb{K}/\mathbb{Q}}(k)\right)^{M+1-m} X_{1,m} \ ; \ m = M, M-1, \ldots, 2 \tag{23}$$

Since $\text{N}_{\mathbb{K}/\mathbb{Q}}(k) \in \mathbb{Q}$, eq. (23) implies that $X_{1,1}, \ldots, X_{1,M}$ (and in an equivalent manner $X_{2,1}, \ldots, X_{2,M}$) must belong simultaneously to one of the following sets $\mathbb{Z}^*$, $\theta\mathbb{Z}^*$ or $\mathcal{O}'_{\mathbb{K}}$.

Following from the structure of $\mathcal{A}$ in eq. (18), a maximum number of two components of X_1 (or X_2) can contain an integer or an integral multiple of θ. For $M = 2$, eq. (23) does not contradict the structure of $\mathcal{A}$ implying that the proposed code is not fully diverse with $M = 2$. Designate by $x_1, \ldots, x_4$ any four elements of the M'-PAM constellation and by i and j two integers verifying $i, j \in \{1, 2\}$ and $i \neq j$. From eq. (18), both entries of X_1 can belong to $\mathbb{Z}^*$ when $b = b'$ and $(a, a') = (x_1 e_i, x_2 e_j)$ (e_m is the m-th column of I_M). In the same way, both entries of X_1 can belong to $\theta\mathbb{Z}^*$ when $a = a'$ and $(b, b') = (x_1 e_i, x_2 e_j)$. Finally, $X_{1,1}$ and $X_{1,2}$ can both be in $\mathcal{O}'_{\mathbb{K}}$ when X_1 takes the form: $X_1 = (x_1 + \theta x_2) e_i + (x_3 + \theta x_4) e_j$.

For $M = 3$, when $X_{1,m} \neq 0$ for $m = 1, \ldots, M$, $X_1 \in \mathcal{A}$ implies that X_1 must belong to the set of all possible permutations of:

$$\mathcal{A}' = \left\{ [x_1, x_2, x_3\theta]^T, [x_1, x_2\theta, x_3\theta]^T, [x_1, x_2\theta, x_3 + x_4\theta]^T \right\} \tag{24}$$

where $x_1, \ldots, x_4 \in \mathbb{Z}^*$. Therefore, a maximum number of 2 components of X_1 can be in $\mathbb{Z}^*$ (or $\theta\mathbb{Z}^*$) at the same time while only one component can belong to $\mathcal{O}'_{\mathbb{K}}$. This is in contradiction with eq. (23) which proves that the proposed code is fully diverse for $M = 3$.

For $M = 4$, a vector $X_i \in \mathcal{A}$ occupies four positions without having any zero entry only if it is a permutation of the vector $(x_1 e_1 + \theta x_2 e_2 + x_3 e_3 + \theta x_4 e_4)$. This implies that there are two components of X_i that belong to $\mathbb{Z}^*$ while the other 2 components belong to $\theta\mathbb{Z}^*$. Such vectors do not verify eq. (23) and, consequently, they do not result in rank deficient matrices ΔC. Considering the last lemma and the cases $M = 3$ and $M = 4$, we conclude that the proposed code achieves full diversity for $n = 2$, $M \geq 3$ and $\forall M'$.

For the general values of n, the reader is referred to Appendix I.

B. Coding Gain

Designate by $\delta_{M,M'}$ the coding gain of the code given in eq. (14) with M-PPM-M'-PAM constellations. We have $\delta_{N,M'} \geq \delta_{M,M'}$ for $N \geq M$ since in this case N-PPM-M'-PAM is obtained by adding new dimensions to the initial M-PPM-M'-PAM signal set. On the other hand, $\delta_{M,N'} \leq \delta_{M,M'}$ for $N' \geq M'$ since M'-PAM is a subset of N'-PAM for $N' \geq M'$. Therefore, unlike the QAM constellations where an increase in the spectral efficiency results in a possible reduction in the coding gain, the coding gain and the spectral efficiency of the hybrid PPM-PAM constellations can be increased by increasing the dimensionality of the considered constellation. Therefore, we can argue that for $M >> 1$, $\delta_{M,M'}$ keeps the same value for all values of M'. This follows from the fact that the multiplication by the matrix Ω simply permutes the coordinates of the information vectors over the M-dimensional signal subspace without introducing any constellation expansion. Formally speaking, the proposed code verifies the following property:

***Proposition** 3.2:* For a system equipped with n transmit antennas with M-PPM-M'-PAM constellations, eq. (14) has a non-vanishing determinant for $M > n^2$ and for all values of M'.

The proof is provided in Appendix II.

C. Shaping and Information Losses

In order to verify the shaping constraint, the basis $\{1, \theta, \ldots, \theta^{n-1}\}$ in eq. (14) can be replaced by a new orthonormal basis $\{v_i\}_{i=1}^n$. In this case, eq. (14) takes the general form:

$$C = \sum_{i=0}^{n-1} \text{Diag}\left((\mathcal{M} \otimes I_M) [\, a_{in+1}^T \cdots a_{(i+1)n}^T \,]^T \right) \Lambda^i \tag{25}$$

For a nM-dimensional vector X, $\text{Diag}(X) = \text{diag}\left([X_1 \cdots X_n]\right)$ where X_i is the M-dimensional vector composed of the j-th components of X for $j = (i-1)M + 1, \ldots, iM$ where $i \in \{1, \ldots, n\}$. The (i, j)-th element of $\mathcal{M}$ is equal to $\sigma^{i-1}(v_j)$ and Λ is the $nM \times nM$ matrix given by:

$$\Lambda = \begin{bmatrix} \mathbf{0}_{M\times M} & I_M & \cdots & \mathbf{0}_{M\times M} \\ \vdots & \mathbf{0}_{M\times M} & \ddots & \vdots \\ \mathbf{0}_{M\times M} & \vdots & \ddots & I_M \\ \Omega & \mathbf{0}_{M\times M} & \cdots & \mathbf{0}_{M\times M} \end{bmatrix} \tag{26}$$

Concerning the choice of the basis $\{v_i\}_{i=1}^n$, the orthonormal basis constructed in [5] can be readily used for all values of n. In the same way, $\mathcal{M}$ can be any fully diverse rotation matrix [19], [20].

***Proposition** 3.3:* The code given in eq. (25) is information lossless for all values of n.

The proof follows from the fact the basis $\{v_i\}_{i=1}^n$ is orthonormal and that the matrix Ω is unitary. A detailed proof is provided in Appendix III.

IV. Construction 2: Achieving full transmit diversity via position combining

In this section, we propose the construction of a new family of ST codes for M-PPM-M'-PAM constellations with the values of M spanning intervals that are potentially different from the ones given in proposition 3.1. In particular, the proposed codes can be applied with any number of transmit antennas for all even values of M. In order to achieve full transmit diversity with these values of M, we propose to keep the same structure of the codewords (as in eq. (14)) and to find an appropriate structure of the $M \times M$ matrix Ω that permits to achieve this diversity order. For a given number of transmit antennas, Ω is constructed as follows:

***Proposition** 4.1:* For n transmit antennas, consider the code given in eq. (14) constructed over $\mathbb{K}(\theta)$ with $\theta = 2\cos(\frac{2\pi}{N})$ and $\varphi(N) = 2n$ (the maximal real subfield of the cyclotomic field with degree n). This code is fully diverse with M-PPM-M'-PAM constellations for all values of M' and for even values of M if:

$$\Omega = I_{M/2} \otimes \begin{bmatrix} \cos(\theta') & \sin(\theta') \\ -\sin(\theta') & \cos(\theta') \end{bmatrix} \tag{27}$$

such that $\theta = \frac{2\pi}{N'}$ and N' is an integer that is relatively prime with N verifying $\varphi(N') \geq 2n$.

Proof: The proof is provided in Appendix IV.

In order to have more insights on the properties of the constructed code, we further investigate the case $n = M = 2$ and $N = 5$. In this case, N' can be chosen to verify proposition 4.1 ($N' = 7$ for example). However, for this special case, we can prove that full diversity can be achieved if the matrix Ω takes the following form:

$$\Omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{28}$$

As in eq. (17), denote by ΔC the difference between two codewords. For $M = n = 2$ and for Ω verifying eq. (28), ΔC can be expressed as:

$$\Delta C = \begin{bmatrix} X_{1,1} & X_{1,2} & \sigma(X_{2,2}) & -\sigma(X_{2,1}) \\ X_{2,1} & X_{2,2} & \sigma(X_{1,1}) & \sigma(X_{1,2}) \end{bmatrix}^T \tag{29}$$

ΔC is rank deficient if $\Delta C_2 = k\Delta C_1$ for a certain value $k \in \mathbb{K}$ where ΔC_i is the i-th column of ΔC. This results in:

$$\begin{aligned} X_{2,1} &= kX_{1,1} = k\sigma(k)X_{2,2} = k^2\sigma(k)X_{1,2} = -k^2(\sigma(k))^2 X_{2,1} \\ &= -\left(\text{N}_{\mathbb{K}/\mathbb{Q}}(k)\right)^2 X_{2,1} \end{aligned} \tag{30}$$

$(1 + (\text{N}_{\mathbb{K}/\mathbb{Q}}(k))^2)X_{2,1} = 0$ implies that $X_{2,1} = 0$ since k is real. From the first row of ΔC, rank$(\Delta C) < 2$ and $X_{2,1} = 0$ $\Rightarrow X_{1,1} = 0$. Considering the third row of ΔC, this implies that $X_{2,2} = 0$ which further results in $X_{1,2} = 0$ from the linear dependence between the elements of the second row of ΔC. Therefore, only the all-zero matrix is rank-deficient and the code is fully diverse for all values of M'.

For the special case of 2-PPM constellations ($M = 2$ and $M' = 1$), the following proposition holds:

***Proposition** 4.2:* If Ω has the structure given in eq. (28), then the code is fully diverse with 2-PPM constellations and for all values of n.

Proof: Designate by $\mathcal{A}$ the set whose elements consist of the difference between two M-dimensional vectors having the structure given in eq. (15). In other words:

$$\mathcal{A} = \{\sum_{i=0}^{n-1}(a_i - a_i')\theta^i \ ; \ a_0, a_0', \ldots, a_{n-1}, a_{n-1}' \in \mathcal{C}\} \tag{31}$$

where $\mathcal{C}$ stands for the PPM-PAM signal set. For binary PPM constellations, $\mathcal{C} = \{[1 \ 0]^T, [0 \ 1]^T\}$. Following from this structure of $\mathcal{C}$, an element A of $\mathcal{A}$ can be written as $A = [a \ \ -a]^T$ where $a \in \mathbb{K}$. For example, with $n = 2$ transmit antennas, consider the following matrix:

$$A = \begin{bmatrix} 0 & 1 & \theta & 1+\theta & 1-\theta \\ 0 & -1 & -\theta & -1-\theta & -1+\theta \end{bmatrix} \tag{32}$$

then, the elements of $\mathcal{A}$ are equal to the columns of the above matrix multiplied by ± 1.

In Appendix V, we show that this particular structure of the elements of $\mathcal{A}$ results in rank$(\Delta C) = n$ for non-zero matrices ΔC and for all values of n. Note that for binary PPM, full diversity can be achieved with the values of Ω respecting proposition 4.1 or with Ω as in eq. (28). It is better to choose the matrix form given in eq. (28) because in this case the multiplication by Ω does not result in an additional constellation extension. Moreover, this choice of Ω results in an additional advantage stated in the next proposition.

***Proposition** 4.3:* If Ω has the structure given in eq. (28), then the code proposed in eq. (14) and the perfect codes [1] have the same coding gain with 2-PPM constellations and for all values of n.

Proof: The proof is given in Appendix VI.

As in section III, the basis $\{1, \dots, \theta^{n-1}\}$ can be replaced by an orthonormal basis resulting in information lossless and shaping efficient codes obtained from the association of eq. (25) with the values of Ω given in eq. (27) or eq. (28). The proof is similar to the one given in Appendix III.

Finally, combining the constructions proposed in this section and in section III permits to achieve full transmit diversity with M-PPM-M'-PAM constellations for all values of M' and for $M \geq 2$, $\{M \geq 2, M \neq 3\}$, $\{M \geq 2, M \neq 3\}$, $\{M \geq 2, M \neq 3, 5\}$ and $\{M \geq 2, M \neq 3, 5\}$ with $n = 2$, 3, 4, 5 and 6 transmit antennas respectively. If a certain values of M verifies proposition 3.1 and proposition 4.1 simultaneously, it is better to choose the matrix form given in eq. (16) that simply permutes the coordinates of the transmitted information vectors rather than combining them.

V. Comparison with codes from cyclic division algebras

In what follows, we compare the modulation specific codes with the totally-real codes constructed from cyclic division algebras for the values of M respecting proposition 3.1 or proposition 4.1.

For the values of M respecting proposition 3.2, construction 1 has a coding gain that does not decrease with the size of the embedded PAM constellation (M'). However, this happens for $M > n^2$ which corresponds to constellations having a large number of dimensions and consequently for systems having a high receiver complexity. For constellations with limited dimensions, the modulation specific codes do not have a non-vanishing determinant (NVD). The advantage of these codes is that they are energy-efficient and information lossless. On the other hand, totally-real codes from CDAs have a non-vanishing determinant. However, this comes at the expense of not having uniform average transmitted energy per antenna. Moreover, these codes are not information lossless codes. For UWB communications, we argue that energy efficiency can be more important than the NVD property for the following reasons:

- Over Rayleigh fading channels, codes with NVD achieve the diversity-multiplexing tradeoff of the underlying channel [21]. However, UWB signals are subject to lognormal fading [8]. For these channels, the asymptotic analysis applied in [21] gives no insights whether or not codes with NVD achieve the D-MG tradeoff over the UWB channels. First, studying the D-MG tradeoff for any number of transmit antennas necessitates the knowledge of the joint probability density function of the eigenvalues of the channel matrix [21], [22]. While these distributions are known in the case of Gaussian random matrices, the channel matrices characterizing the propagation of UWB signals (in eq. (12)) do not lend themselves to an analytical solution. Second, because it was proven in [9] that an infinite diversity order can be reached for asymptotic values of the SNR even with single-antenna systems. Over the UWB channels, the absence of a direct connection between the NVD property and the achievable tradeoffs reduces the importance of the NVD property.
- In [21], the tradeoff is achieved with a family of codes whose spectral efficiency scales with the SNR. For QAM constellations, this reduces the coding gain unless when the code has a NVD. On the other hand, at high SNRs, the spectral efficiency of M-PPM-M'-PAM can be increased by extending the constellation in the time domain (increasing M) rather than the amplitude domain (increasing M'). For construction 1, this extension does not introduce a decrease in the coding gain since the dimensionality of the signal subspace is increased with no additional constellation expansion.
- High order PAM constellations have a very poor performance [23]. This is why, the highest order PAM constellation used in the literature of the UWB was the 2-PAM constellation.

Finally, the modulation-specific codes verify a large number of constraints. 1) They are totally-real. 2) They achieve full rate and full diversity. 3) They introduce no shaping losses. 4) They are information lossless. 5) They are energy efficient. 6) They achieve the best known coding gains (those of the complex-valued perfect codes) in the following cases:

- M-PPM-M'-PAM constellations for all values of M' and for $M > n^2$ with any number of transmit antennas n (from proposition 3.2).
- 2-PPM constellations with any number of transmit antennas (from proposition 4.3).
- 2-PPM-2-PAM, 2-PPM-4-PAM and 2-PPM-8-PAM constellations with $n = 2$ transmit antennas (with the value of Ω given in eq. (28)).
- M-PPM constellations for all values of $M \geq 3$ and for $n = 2$ transmit antennas (construction 1).
- M-PPM-2-PAM (bi-orthogonal) constellations and M-PPM-4-PAM for $M \geq 4$ and for $n = 2$ transmit antennas (construction 1).

The third case is verified by computer simulations. The last two cases follow from the numerical evaluation of the coding gain of the code in eq. (25) associated with the value of Ω in eq. (16) with M-PPM and M-PPM-M'-PAM ($M' \in \{2, 4\}$) for $M = 3$ and $M = 4$ respectively. Since the coding gain of construction 1 increases with M, this proves the last two conditions.

VI. Simulations and results

The pulse waveform $w(t)$ is chosen to be the second derivative of the Gaussian pulse with a duration of 0.5 ns and we fix $\delta = T_w = 0.5$ ns. N_f has no effect on the performance in single user situations and it is fixed to 1. At a first time, the PQ channels between the different antennas are generated

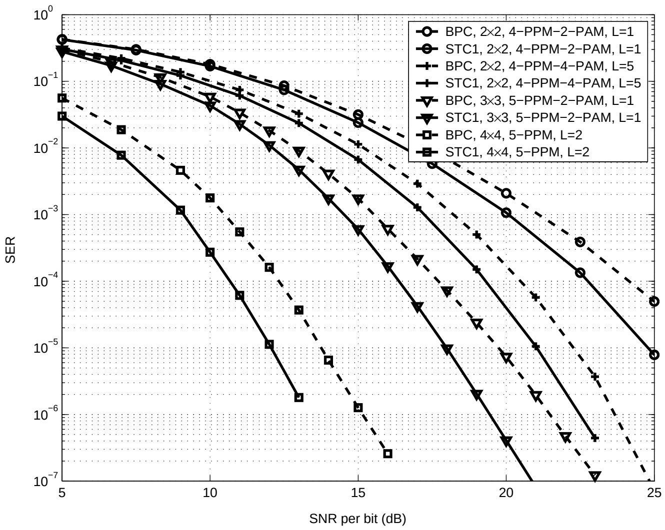

Fig. 2. The best previously known code (BPC) [5] versus construction 1 (STC1) on CM2.

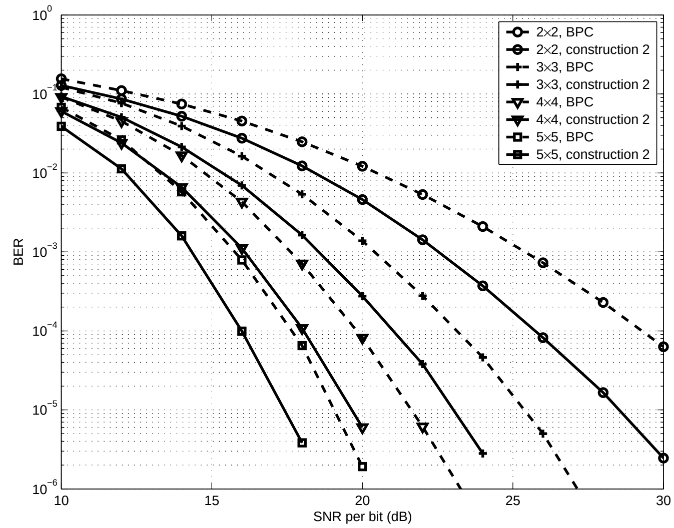

Fig. 3. The best previously known code (BPC) [5] versus construction 2 on CM2 with 2-PPM and $L = 1$.

independently according to the IEEE 802.15.3a channel model recommendation [8]. The channel is held constant over one transmission block and is allowed to change from one block to another. The frame duration T_f is chosen to be $T_f = 100$ ns which is larger than the maximum delay spread of the channel models CM1 and CM2 that correspond to line-of-sight and non-line-of-sight conditions respectively [8]. For simplicity, we assume that the relative time delays between the signals received at the different antennas ($\varepsilon_{q,p}$ in eq. (7)) are negligible. In this way, simulations highlight the diversity and multiplexing advantages of the proposed schemes independently from the relative orientations and positions of the transmit and receive arrays. The sphere decoder [24] is used for detection.

Fig. 2 compares the best previously known codes [5] (that are based on totally-real CDAs) with the modulation-specific codes using M-PPM-M'-PAM constellations. The values of M are chosen to verify proposition 3.1 and therefore Ω can take the value given in eq. (16). Results show that the modulation-specific codes outperform codes from totally-real cyclic division algebras without the need of any pulse repetitions or additional number of matched filters. This shows the importance of having energy-efficient codes. A gain of about 1 dB is observed at an error rate of 10^{-3} using three transmit antennas and 5-PPM-2-PAM constellations. This gain increases to 1.5 dB with four transmit antennas and 5-PPM.

In Fig. 3, we consider 2-PPM constellations on CM2. The receiver is equipped with a 1-finger PRake and we fix $P = Q = n$ where $n = 2, \ldots, 5$. Following from proposition 4.2, Ω can take the form given in eq. (28). In this case, the association of construction 2 with 2-PPM achieves the best known coding gain (proposition 4.3). As a result, performance gains in the order of 2 dB can be obtained with respect to [5] with 3 transmit antennas at a BER of 10^{-3}.

In Fig. 4, we show the performance of the proposed ST codes over the space-variant UWB channel model proposed in [25], [26]. In this case, the relative delays between the elements of the antenna arrays are provided by this model ($\varepsilon_{q,p}$ in eq. (7)). We fix, $P = 3$, $Q = 1$ and $L = 5$ with 4-PPM constellations. For the modulation-specific codes, Ω has the structure given in eq. (27) with $N' = 9$ (since $N = 7$). We consider profile 3 that corresponds to an "office-to-office" scenario. We compare the cases where the distance d between the elements of the transmit antenna array is equal to 5 cm and 15 cm respectively. Results show that even though construction 2 introduces an additional constellation extension, its good energetic distribution results in a performance gain of about 1.2 dB with respect to totally-real codes from CDAs at a SER of 10^{-3}. The slopes of the error curves of the ST coded systems highlight the enhanced diversity order with respect to the uncoded MIMO systems. Closely separated antenna arrays ($d = 5$ cm) result in a performance loss of about 0.6 dB with respect to arrays whose inter-antenna separation is equal to 15 cm. However, this performance loss is not associated with any degradation in the diversity order and the same diversity advantage can be obtained for both inter-antenna separations.

In Fig. 5, the proposed modulation-specific codes are compared with the perfect codes for the values of M and M' that result in the best known coding gain (refer to section V). For comparison reasons, and even though it may seem practically infeasible, the UWB receivers are supposed to be equipped with IQ front ends. The results show the high performance levels of the proposed modulation-specific schemes. Even though they are real-valued, they show exactly the same performance as the best known codes.

In Fig. 6, we compare the best known totally-real codes from CDAs [5] and the modulation-specific code (construction 2) with 2 transmit antennas, 1 receive antenna and 2-PPM-M'-PAM constellations for $M' \in \{4, 16, 64\}$ over CM1 using a 5-finger PRake. It was previously shown that the code based on CDAs has a non-vanishing determinant while the modulation-specific codes are information lossless. For the latter codes, there is no explicit expression of the coding gain given that the determinants of the codewords are not rational integers.

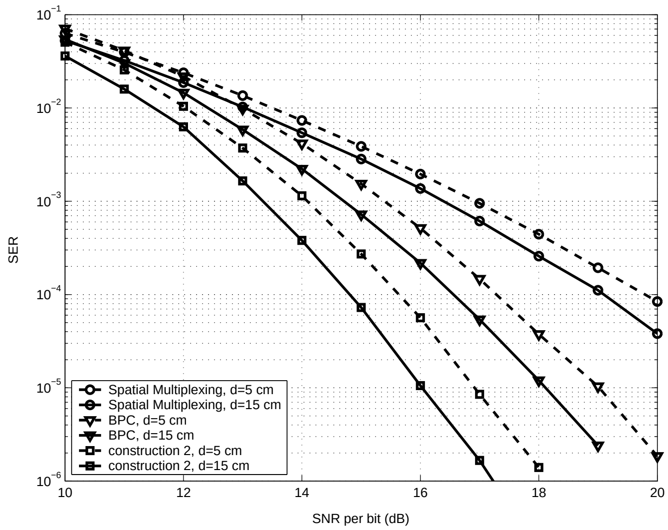

Fig. 4. The best previously known code (BPC) [5] versus construction 2 on the Kunisch-Pamp channel model (profile 3) using 4-PPM, $P = 3$, $Q = 1$ and $L = 5$. d corresponds to the inter-antenna separation.

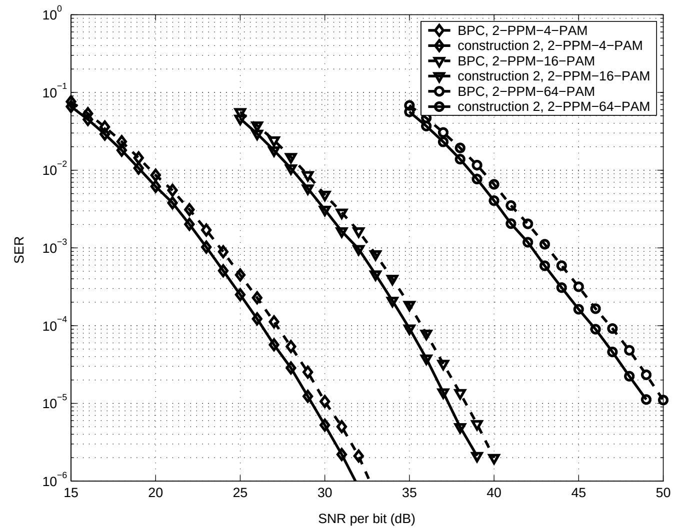

Fig. 6. The best previously known code (BPC) [5] versus construction 2 with 2 transmit antennas, 1 receive antenna and 2-PPM-M'-PAM constellations over CM1. A 5-finger PRake is used.

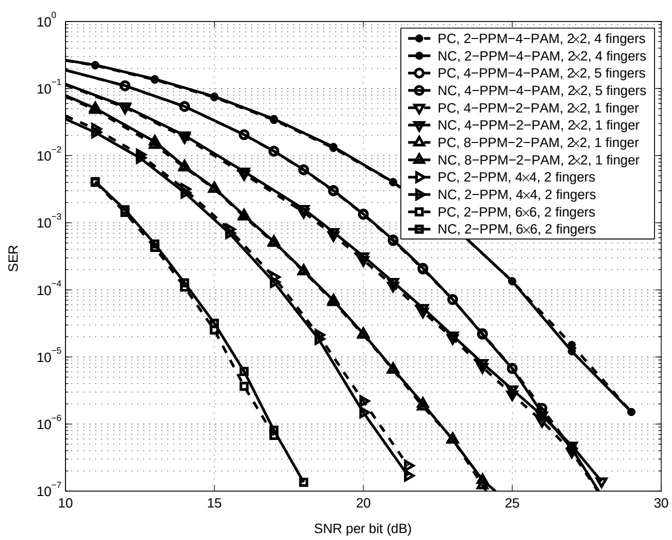

Fig. 5. The new modulation-specific codes (NC) versus the perfect codes (PC) on CM2.

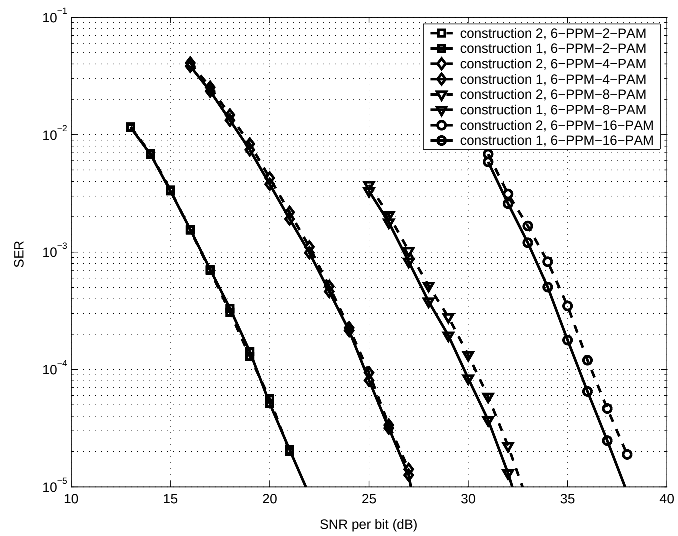

Fig. 7. Construction 1 versus construction 2 with 3 transmit antennas, 3 receive antenna and 6-PPM-M'-PAM constellations over CM2.

We might imagine that codes from CDAs will outperform the modulation-specific codes for large values of M' since the coding gain of the former remains constant. However, the results in Fig. 6 show the superiority of the modulation-specific codes even at very high spectral efficiencies. Note also that for large values of M', the performance losses of both codes are very important even at very large values of the SNR. Therefore, practical single-antenna and multi-antenna UWB systems might not use these high order embedded PAM constellations.

In Fig. 7, we compare construction 1 and construction 2 with M-PPM-M'-PAM constellations for the values of M that verify proposition 3.1 and proposition 4.1 simultaneously. In particular, we perform simulations over CM2 with $P = 3$ transmit antennas, 3 receive antennas and a 1-finger Rake for $M = 6$ and $M' \in \{2, 4, 8, 16\}$. Given the high dimensionality of the transmitted constellation ($P^2M = 54$), we apply the algorithm proposed in [27] for detection. This decoding algorithm is based on the Schnorr-Euchner enumeration strategy and a Fano-like metric bias is introduced in order to enhance the convergence time. Moreover, the MMSE-DFE preprocessing is performed [28]. As expected, construction 1 outperforms construction 2 and the simulation results show that the performance difference between these two constructions becomes significant for large values of M'. While construction 2 shows practically the same performance as construction 1 for $M' = 2, 4$, the former construction results in a performance loss in the order of 0.5 dB for $M' = 8, 16$.

VII. CONCLUSION

We investigated the problem of constructing ST coding schemes suitable for UWB systems using M-PPM-M'-PAM as a candidate solution for very high data rate systems. The novel coding schemes are based on "position permutation" and "position combining". They solve the problem of the nonexistence of energy-balanced, information lossless and totally-real constructions and they outperform the classical approach of constructing totally-real ST codes based on cyclic division algebras. In some situations, the modulation-specific codes are capable in delivering the same performance levels as the complex-valued prefect codes.

APPENDIX I
DIVERSITY ORDER OF CONSTRUCTION 1

In this appendix, we determine the dimensions of the hybrid PPM-PAM constellation for which the code in eq. (14) is fully diverse for a given value of n.

Designate by $\Delta C(X_1,\ldots,X_n)$ the difference between two codewords having the structure given in eq. (14). X_i is a M-dimensional vector that can be written as: $X_i = x_i^{(1)} - x_i^{(2)}$ where:

$$x_i^{(k)} = \sum_{j=0}^{n-1} x_{i,j}^{(k)} \theta^j \ ; \ k = 1,2 \tag{33}$$

where the vectors $x_{i,j}^{(k)}$ belong to the M-PPM-M'-PAM constellation $\mathcal{C}$ given in eq. (3) for $k = 1,2$, $i = 1,\ldots,n$ and $j = 0,\ldots,n-1$. In other words, the vectors $X_1,\ldots,X_n$ belong to the set $\mathcal{A}$ given by:

$$\mathcal{A} = \{\sum_{j=0}^{n-1}(a_j - a'_j)\theta^j \mid a_0, a'_0, \ldots, a_{n-1}, a'_{n-1} \in \mathcal{C}\} \tag{34}$$

The proof now consists of finding the set of values of M for which the following relation holds: $\text{rank}(\Delta C(X_1,\ldots,X_n)) = n$ for $(X_1,\ldots,X_n) \in \mathcal{A}^n \backslash \{(\mathbf{0}_M,\ldots,\mathbf{0}_M)\}$.

For an element $k \in \mathbb{K} \mid k = \sum_{j=0}^{n-1} k_j \theta^j$, $k_j \in \mathbb{Q}$ will be referred to as the j-th coordinate of k. In eq. (34), $a_0, a'_0, \ldots, a_{n-1}, a'_{n-1}$ are integer multiples of the columns of the $M \times M$ identity matrix. Therefore, elements of $\mathcal{A}$ have the property that at most two of their components can have their j-th coordinates different from zero for $j = 0,\ldots,n-1$.

$\Delta C(X_1,\ldots,X_n)$ will be referred to as ΔC for simplicity. $\text{rank}(\Delta C) < n$ if $\exists k_1,\ldots,k_{n-1}$ such that:

$$\Delta C_n = \sum_{i=1}^{n-1} k_i \Delta C_i \tag{35}$$

where ΔC_i is the i-th column of ΔC. Moreover, $k_1,\ldots,k_{n-1} \in \mathbb{K}$ since all of the elements of ΔC are in $\mathbb{K}$.

Proposition *1.1:* Equation (35) is verified if the vectors $X_1,\ldots,X_n$ verify the equation:

$$\mathcal{M}X_i = (\sum_{j=0}^{n-1} \lambda_j \Omega^j) X_i = \mathbf{0}_M \ ; \ i = 1,\ldots,n \tag{36}$$

where Ω is the permutation matrix given in eq. (16). $\lambda_1,\ldots,\lambda_{n-1} \in \mathbb{Q}$ and $\lambda_0 = 1$ resulting in $\mathcal{M} \in \mathbb{Q}^{M\times M}$.

Proof: We first give an example with $n = 3$. Inserting eq. (14) in eq. (35), we obtain:

$$\begin{aligned} X_3 &= k_1X_1 + k_2X_2 \\ \sigma(X_2) &= \Omega k_1 \sigma(X_3) + k_2\sigma(X_1) \\ \sigma^2(X_1) &= \Omega k_1 \sigma^2(X_2) + \Omega k_2 \sigma^2(X_3) \end{aligned}$$

Since the entries of Ω are rational numbers, then $\sigma(\Omega) = \Omega$ and the above equations can be written as:

$$\begin{aligned} X_3 &= k_1X_1 + k_2X_2 \\ X_2 &= \Omega\sigma^2(k_1)X_3 + \sigma^2(k_2)X_1 \\ X_1 &= \Omega\sigma(k_1)X_2 + \Omega\sigma(k_2)X_3 \end{aligned}$$

Solving these equations, we conclude that the vectors X_i for $i = 1,\ldots,3$ must verify:

$$\left(I_M + \lambda_1\Omega + \lambda_2\Omega^2\right) X_i = \mathbf{0}_M \tag{37}$$

$$\lambda_1 = -\left(\text{Tr}_{\mathbb{K}/\mathbb{Q}}(\sigma^2(k_1)k_2) + \text{N}_{\mathbb{K}/\mathbb{Q}}(k_2)\right) \ ; \ \lambda_2 = -\text{N}_{\mathbb{K}/\mathbb{Q}}(k_1) \tag{38}$$

In the same way, with $n = 4$ antennas, we have:

$$\left(I_M + \lambda_1\Omega + \lambda_2\Omega^2 + \lambda_3\Omega^3\right) X_i = \mathbf{0}_M \tag{39}$$

$$\begin{aligned} \lambda_1 &= -\text{N}_{\mathbb{K}/\mathbb{Q}}(k_3) - \text{Tr}_{\mathbb{K}/\mathbb{Q}}(k_1\sigma(k_3)) - \text{Tr}_{\mathbb{K}/\mathbb{Q}}(k_2\sigma(k_3)\sigma^2(k_3)) \\ &\quad - \frac{1}{2}\text{Tr}_{\mathbb{K}/\mathbb{Q}}(k_2\sigma^2(k_2)) \\ \lambda_2 &= -\text{Tr}_{\mathbb{K}/\mathbb{Q}}(k_2\sigma(k_2)\sigma^3(k_1)\sigma^2(k_3)) - \text{Tr}_{\mathbb{K}/\mathbb{Q}}(k_2\sigma^2(k_1)\sigma^3(k_1)) \\ &\quad + \text{N}_{\mathbb{K}/\mathbb{Q}}(k_2) + \frac{1}{2}\text{Tr}_{\mathbb{K}/\mathbb{Q}}(\sigma(k_1)\sigma^3(k_1)k_3\sigma^2(k_3)) \\ \lambda_3 &= -\text{N}_{\mathbb{K}/\mathbb{Q}}(k_1) \end{aligned}$$

We return now to the general case. The columns of ΔC are considered as n-dimensional vectors composed of the parametric variables Ω, $X_1,\ldots,X_n$ and their conjugates. Therefore, eq. (35) results in n linear equations. Taking the $(n-i)$-th conjugate of the i-th equation for $i = 2,\ldots,n$, eq. (35) can be written as:

$$R \begin{bmatrix} X_1 \\ \vdots \\ X_{n-1} \\ X_n \end{bmatrix} = \mathbf{0}_{Mn} \tag{40}$$

where the matrix R is given by:

$$\begin{bmatrix} k_1 & k_2 & \cdots & k_{n-1} & -1 \\ \sigma^{n-1}(k_2) & \ddots & \sigma^{n-1}(k_{n-1}) & -1 & \Omega\sigma^{n-1}(k_1) \\ \vdots & \ddots & \ddots & \Omega\sigma^{n-2}(k_1) & \Omega\sigma^{n-2}(k_2) \\ \sigma^2(k_{n-1}) & -1 & \ddots & \ddots & \vdots \\ -1 & \Omega\sigma(k_1) & \Omega\sigma(k_2) & \cdots & \Omega\sigma(k_{n-1}) \end{bmatrix} \tag{41}$$

The last equation is verified if:

$$\det(R)X_i = \mathbf{0}_M \ ; \ i = 1,\ldots,n \tag{42}$$

when calculating the determinant, R is considered as a $n \times n$ matrix and Ω is taken as a parameter (rather than a $M \times M$ matrix). By rearranging the rows of R, we obtain:

$$\det(R) = \det(\underbrace{[R_n^T, R_{n-1}^T, \ldots, R_2^T, R_1^T]^T}_{R'}) = \det(R') \tag{43}$$

where R_i is the i-th row of i. R' has the same structure as the matrix expression of the elements of a division algebra. In particular:

$$R' = C\left(-1, \sigma(k_1), \ldots, \sigma(k_{n-1})\right) \tag{44}$$

where the expression of C is given in eq. (1) (by replacing the variable γ with Ω). From [11], the coefficients of the powers of Ω in $\det(C)$ are linear combinations of norms and traces that belong to $\mathbb{Q}$. Consequently, parameterized by Ω, $\det(R)$ is a polynomial with rational coefficients having a degree of $n-1$:

$$\det(R) = \sum_{i=0}^{n-1} \lambda_i \Omega^i \ ; \ \lambda_0, \ldots, \lambda_{n-1} \in \mathbb{Q} \tag{45}$$

Moreover, it can be easily shown that $\lambda_0 = 1$ and $\lambda_{n-1} = -\mathrm{N}_{\mathbb{K}/\mathbb{Q}}(k_1)$. Finally, replacing the parameter Ω by its corresponding matrix form, we obtain eq. (36).

Consider the case $M \geq n$. The matrix $\mathcal{M}$ in eq. (36) takes the form:

$$\mathcal{M} = \begin{bmatrix} 1 & 0 & \cdots & 0 & \lambda_{n-1} & \cdots & \lambda_1 \\ \lambda_1 & 1 & 0 & \cdots & 0 & \ddots & \vdots \\ \lambda_2 & \lambda_1 & 1 & 0 & \cdots & 0 & \lambda_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \lambda_{n-1} & \cdots & \lambda_1 & 1 & 0 & \vdots \\ \vdots & 0 & \lambda_{n-1} & \cdots & \lambda_1 & 1 & 0 \\ 0 & \cdots & 0 & \lambda_{n-1} & \cdots & \lambda_1 & 1 \end{bmatrix} \tag{46}$$

where the number of zero entries in each row of $\mathcal{M}$ is equal to $M-n$.

From eq. (46), it can be seen that the matrix composed of the first $(M-n+1)$ rows and the first $(M-n+1)$ columns of $\mathcal{M}$ is a lower triangular matrix whose diagonal elements are all equal to 1. Therefore, its determinant is equal to 1 and consequently it has a rank of $M-n+1$. Designate by r the rank of $\mathcal{M}$. Therefore, $r \geq M-(n-1)$. In an equivalent manner:

$$r = M - k \ ; \ k \in \{0, 1, \ldots, n-1\} \tag{47}$$

Since eq. (36) is valid for all values of i, we fix $Y = X_i$. Denote by Y_m the m-th component of Y for $m = 1, \ldots, M$. The j-th coordinate of Y_m is denoted by $Y_{m,j}$ for $j = 0, \ldots, n-1$.

For $k = 0$, $\mathcal{M}Y = \mathbf{0}_M \Rightarrow Y = \mathbf{0}_M$. Therefore, the only matrix ΔC that is rank deficient is the all-zero matrix.

For $k = 1$, the M components of Y can be determined from a single parameter t. Without loss of generality, we take $t = Y_M$. In this case, $Y_m = \beta_m t$ with $\beta_m \in \mathbb{Q}^*$ for $m = 1, \ldots, M-1$ since $\mathcal{M} \in \mathbb{Q}^{M \times M}$. $t = 0 \Rightarrow Y = \mathbf{0}_M$. In other words, if one of the components of Y is equal to zero, then $k = 1$ and $\mathcal{M}Y = \mathbf{0}_M$ imply that $Y = \mathbf{0}_M$. For nonzero vectors, denote by $Y_{M,j}$ the first non-zero coordinate of Y_M for a certain value of $j \in \{0, \ldots, n-1\}$. Since $\beta_m \in \mathbb{Q}^*$, all the components of Y will have their j-th coordinates different from zero. Therefore, Y will not belong to $\mathcal{A}$ when $M \geq 3$ since, from eq. (34), any element of $\mathcal{A}$ can have a maximum number of two components whose j-th coordinates are different from zero $\forall j$. Therefore, the only vector of $\mathcal{A}$ that verifies $\mathcal{M}Y = \mathbf{0}_M$ when $r = M-1$ is $Y = \mathbf{0}_M$. As a conclusion, when $k = 1$, the code is fully diverse for:

$$M \geq \max\{3, n\} \tag{48}$$

as a special case, when $n = 2$, $k \in \{0, 1\}$ and the code is fully diverse for $M \geq 3$.

For $k > 1$, the components of Y can be parameterized by k parameters. Without loss of generality, these parameters are taken to be equal to $Y_{M-k+1}, \ldots, Y_M$. The other r parameters can be expressed as:

$$Y_m = \sum_{j=1}^{k} \beta_m^{(j)} Y_{M-k+j} \ ; \ m = 1, \ldots, r \tag{49}$$

where $\beta_m^{(1)}, \ldots, \beta_m^{(k)}$ can not be equal to zero simultaneously for $m = 1, \ldots, r$. We will consider the two following cases.

Case 1: Suppose that there is only one element among $Y_{M-k+1}, \ldots, Y_M$ that has its j-th coordinate different from zero for a given value of j. In other words, there are no two or more elements that have the same coordinate taking a nonzero value. Note that this can always happen since $k \leq n-1$. Denote by j_l the first nonzero coordinate of Y_{M-k+l}. Among the first r components of Y there are r_l components whose j_l-th coordinates are not equal to zero with $1 \leq r_l \leq r$ for $l = 1, \ldots, k$. Along with Y_{M-k+l}, this results in a total of $r_l + 1$ components of Y having their j_l-th components different from zero. The k coefficients $\beta_m^{(1)}, \ldots, \beta_m^{(k)}$ can not be all equal to zero simultaneously for a given value of m verifying $1 \leq m \leq r$. This implies that there are at least r nonzero coefficients among $\beta_1^{(1)}, \ldots, \beta_1^{(k)}, \ldots, \beta_r^{(1)}, \ldots, \beta_r^{(k)}$. This results in: $\sum_{l=1}^{k} r_l \geq r$.

Y will fall outside of $\mathcal{A}$ (and consequently the code will be fully diverse) whenever $\exists l \in \{1, \ldots, k\} \mid r_l + 1 > 2$. In this case, the maximum size of the constellation for which Y remains in $\mathcal{A}$ corresponds to the case when $r_l = 1$ for $l = 1, \ldots, k$. Now, $\sum_{l=1}^{k} r_l \geq r$ implies that $k \geq r$. Given that the maximal value of k is equal to $n-1$, Y falls outside of $\mathcal{A}$ when $r > k$ implying that:

$$M \geq 2n - 1 \tag{50}$$

When considering the second case, we limit ourselves to the constellation sizes that respect eq. (50) (or equivalently $r > k$).

Case 2: Following from the structure of $\mathcal{A}$, the complementary of case 1 takes place when there exist two distinct values $k', k'' \in \{1, \ldots, k\}$ such that $Y_{M-k+k'}$ and $Y_{M-k+k''}$ have their j-th coordinates different from zero for a certain value of $j \in \{0, \ldots, n-1\}$. Designate by Υ the $k \times k$ matrix whose (i,j)-th element is equal to $\beta_i^{(j)}$ for $i, j = 1, \ldots, k$. Υ describes the linear dependence of the first k components of Y on $Y_{M-k+1}, \ldots, Y_M$. Given the lower triangular structure of the matrix composed of the first k rows and k columns of $\mathcal{M}$ and since $r > k$, this implies that Υ has a rank k.

Among the first k components of Y, designate by $r_1 \leq k$ the maximum number of components having their j-th components equal to zero simultaneously. Since the nonzero j-th coordinates are contained uniquely in $Y_{M-k+k'}$ and $Y_{M-k+k''}$, this implies that there exists r_1 indexes $I(1),\dots,I(r_1)$ such that:

$$\beta_{I(i)}^{(k')} Y_{M-k+k',j} + \beta_{I(i)}^{(k'')} Y_{M-k+k'',j} = 0 \quad ; \quad i = 1,\dots,r_1 \tag{51}$$

From eq. (51), $r_1 = k$ implies that the k'-th and the k''-th columns of Υ are linearly dependent which is in contradiction with the fact that Υ has a full rank. Consequently $r_1 < k$ and among the first r components of Y there is a minimum of $r-(k-1)$ components whose j-th coordinates are different from zero. Along with $Y_{M-k+k'}$ and $Y_{M-k+k''}$, this will result in a total of $r-(k-3)$ components of Y having nonzero j-th coordinates. Y will fall outside of $\mathcal{A}$ whenever $r-(k-3) > 2$ implying that $M > 2k-1$ or equivalently $M \geq 2(n-1)$. Therefore, the values of M respecting eq. (50) will result in fully diverse codes in this second case.

Form eq. (48) and eq. (50), we conclude that the proposed code is fully diverse with n transmit antennas if $M \geq 2n-1$.

From eq. (46), $\mathcal{M}$ is a circulant matrix that can be expressed as:

$$\mathcal{M} = \sum_{i=1}^{M} \alpha_i \Omega^i \tag{52}$$

where $\alpha_M = \lambda_0 = 1$ (since $\Omega^M = I_M$), $\alpha_i = \lambda_i$ for $i = 1,\dots,n-1$ and $\alpha_i = 0$ for $i = n,\dots,M-1$. From [29], the eigenvalues of $\mathcal{M}$ can be expressed as:

$$\mu_k = \sum_{i=0}^{M-1} \omega_M^{ki} \lambda_{M-i} \quad ; \quad k = 0,\dots,M-1 \tag{53}$$

where $\omega_M = \exp\left(\frac{2\pi i}{M}\right)$ is the M-th root of unity.

Any subset composed of n' distinct elements of $\{1,\omega_M,\dots,\omega_M^{M-1}\}$ forms a free set over $\mathbb{Q}$ if $2 \leq n' \leq \varphi(M)$ where $\varphi(.)$ is the Euler's function. Consider the case where $\varphi(M) \geq n$. In this case, $\mu_k \neq 0$ for $k = 1,\dots,M-1$ since $\lambda_0,\dots,\lambda_{n-1}$ multiply different elements of the set $\{1,\dots,\omega_M^{M-1}\}$ and since $\lambda_0 = 1$. Only $\mu_0 = \sum_{i=0}^{n-1}\lambda_i$ can be equal to zero. Therefore, $r = \text{rank}(\mathcal{M}) \geq M-1$. From what preceded, $\mathcal{M}Y = \mathbf{0}_M$; $Y \in \mathcal{A}$; $r \in \{M, M-1\} \Rightarrow Y = \mathbf{0}_M$ for $M \geq \max\{3,n\}$. The condition $M \geq \max\{3,n\}$ is verified by the values of M that respect $\varphi(M) \geq n$ (since $M > \varphi(M)\ \forall M$). This implies that the code is fully diverse with all values of M respecting $\varphi(M) \geq n$.

Finally, the proposed scheme achieves full transmit diversity with n transmit antennas and M-PPM-M'-PAM constellations if:

$$M \in \{m \mid m \geq 2n-1\} \cup \{m \mid \varphi(m) \geq n\} \tag{54}$$

Note that the above proof does not depend on M'. Therefore, full diversity is achieved for the values of M respecting eq. (54) and for all values of M'.

APPENDIX II
CODING GAIN OF CONSTRUCTION 1

Consider the codeword C given in eq. (14). Denote by $C^{(i)}$ the $M \times n$ matrix composed of the rows $iM+1,\dots,(i+1)M$ of C for $i = 0,\dots,n-1$. The j-th row of $C^{(i)}$ is denoted by $C_j^{(i)}$. In the same way, $X_{i,m}$ denotes the m-th component of the M-dimensional vector X_i for $m = 1,\dots,M$ and $i = 1,\dots,n$. For notational simplicity, we set $\sigma^n(x) = x^{(n)}$. From eq. (14), $C^{(i)}$ takes the following form:

$$C^{(i)} = \begin{bmatrix} X_{n-i+1,\pi(1)}^{(i)} & \cdots & X_{n,\pi(1)}^{(i)} & X_{1,1}^{(i)} & \cdots & X_{n-i,1}^{(i)} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ X_{n-i+1,\pi(M)}^{(i)} & \cdots & X_{n,\pi(M)}^{(i)} & X_{1,M}^{(i)} & \cdots & X_{n-i,M}^{(i)} \end{bmatrix} \tag{55}$$

where $\pi^k(.)$ stands for the cyclic permutation of order k given by:

$$\pi^k(i) = (i+k-1) \mod M + 1 \tag{56}$$

Let $\mathcal{S} = \{X_1,\dots,X_n\}$ be the set composed of n vectors drawn randomly from the set $\mathcal{A}$ given in eq. (34). For $M > n^2$, among the n vectors of $\mathcal{S}$, there exists $n-1$ vectors whose m-th coordinates are all equal to zero for a certain value of $m \in \{1,\dots,M\}$. This follows from the structure of $\mathcal{A}$ whose elements can contain a maximum number of $2n$ nonzero components. In what follows, we show that the code has a non-vanishing determinant for $M > n^2$.

Without loss of generality, suppose that $X_{1,m} = \cdots = X_{n-1,m} = 0$ for a given value of $m \in \{1,\dots,M\}$.

***Proposition** 2.1:* $X_{1,m} = \cdots = X_{n-1,m} = 0$ implies that $\det(C^TC) \geq 1$ unless $X_{n,m} = 0$.

Proof: $\det(C^TC)$ verifies the following relation:

$$\begin{aligned} &\det(C^TC) \\ &= \sum_{i_1=1}^{nM}\sum_{i_2=i_1+1}^{nM}\cdots\sum_{i_n=i_{n-1}+1}^{nM}\left(\det\left(\left[C_{i_1}^T\ \cdots\ C_{i_n}^T\right]^T\right)\right)^2 \\ &\geq \sum_{i_1=1}^{M}\sum_{i_2=M+1}^{2M}\cdots\sum_{i_n=(n-1)M+1}^{nM}\left(\det\left(\left[C_{i_1}^T\ \cdots\ C_{i_n}^T\right]^T\right)\right)^2 \\ &= \sum_{i_1=1}^{M}\sum_{i_2=1}^{M}\cdots\sum_{i_n=1}^{M}\left(\det\left(\left[(C_{i_1}^{(0)})^T\ \cdots\ (C_{i_n}^{(n-1)})^T\right]^T\right)\right)^2 \end{aligned} \tag{57}$$

Construct the $n \times n$ matrix $\mathcal{C}'$ as:

$$\begin{aligned} C' &= \begin{bmatrix} \left(C_{\pi^{-1}(m)}^{(1)}\right)^T & \cdots & \left(C_{\pi^{-1}(m)}^{(n-1)}\right)^T & \left(C_m^{(0)}\right)^T \end{bmatrix}^T \\ &= \begin{bmatrix} X_{n,m}^{(1)} & X_{1,\pi^{(-1)}(m)}^{(1)} & \cdots & \cdots & X_{n-1,\pi^{(-1)}(m)}^{(1)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ X_{2,m}^{(n-1)} & \cdots & \cdots & X_{n,m}^{(n-1)} & X_{1,\pi^{-1}(m)}^{(n-1)} \\ X_{1,m} & \cdots & \cdots & X_{n-1,m} & X_{n,m} \end{bmatrix} \end{aligned} \tag{58}$$

Since $X_{1,m} = \cdots = X_{n-1,m} = 0$, $\mathcal{C}'$ is upper triangular. From eq. (57):

$$\det\left(C^TC\right) \geq \left(\det\left(C'\right)\right)^2 = \left(\text{N}_{\mathbb{K}/\mathbb{Q}}(X_{n,m})\right)^2 \tag{59}$$

Since, $\text{N}_{\mathbb{K}/\mathbb{Q}}(X_{n,m}) \in \mathbb{Z}^*$ for $X_{n,m} \neq 0$, this completes the proof of proposition 2.1.

***Proposition** 2.2:* $X_{1,m} = \cdots = X_{n-1,m} = 0$ implies that $\det(C^TC) \geq 1$ unless $X_{n,\pi(m)} = 0$.

$$C' = \begin{bmatrix} C_m^{(f(0))} \\ \vdots \\ C_m^{(n-1)} \\ C_{\pi(m)}^{(0)} \\ \vdots \\ C_{\pi(m)}^{(f(n-1))} \end{bmatrix} = \begin{bmatrix} X_{n',\pi(m)}^{(f(0))} & \cdots & \cdots & X_{n,\pi(m)}^{(f(0))} & X_{1,m}^{(f(0))} & \cdots & X_{n'-1,m}^{(f(0))} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ X_{2,\pi(m)}^{(n-1)} & \cdots & X_{n',\pi(m)}^{(n-1)} & X_{n'+1,\pi(m)}^{(n-1)} & \cdots & X_{n,\pi(m)}^{(n-1)} & X_{1,m}^{(n-1)} \\ X_{1,\pi(m)} & \cdots & \cdots & X_{n',\pi(m)} & X_{n'+1,\pi(m)} & \cdots & X_{n,\pi(m)} \\ \vdots & \ddots & \ddots & \ddots & \ddots\ddots & \ddots & \vdots \\ X_{n'+1,\pi^2(m)}^{(f(n-1))} & \cdots & X_{n,\pi^2(m)}^{(f(n-1))} & X_{1,\pi(m)}^{(f(n-1))} & \cdots & \cdots & X_{n',\pi(m)}^{(f(n-1))} \end{bmatrix}$$

Proof: Construct the matrix C' as:

$$C' = \left[\begin{array}{cccc} \left(C_m^{(1)}\right)^T & \cdots & \left(C_m^{(n-1)}\right)^T & \left(C_{\pi(m)}^{(0)}\right)^T \end{array} \right]^T$$
$$= \begin{bmatrix} X_{n,\pi(m)}^{(1)} & X_{1,m}^{(1)} & \cdots & \cdots & X_{n-1,m}^{(1)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ X_{2,\pi(m)}^{(n-1)} & \cdots & X_{n-1,\pi(m)}^{(n-1)} & X_{n,\pi(m)}^{(n-1)} & X_{1,m}^{(n-1)} \\ X_{1,\pi(m)} & \cdots & \cdots & X_{n-1,\pi(m)} & X_{n,\pi(m)} \end{bmatrix} \tag{60}$$

Since $X_{1,m} = \cdots = X_{n-1,m} = 0$, $\mathcal{C}'$ is lower triangular. From eq. (57):

$$\det\left(C^T C\right) \geq \left(\det\left(C'\right)\right)^2 = \left(\mathrm{N}_{\mathbb{K}/\mathbb{Q}}(X_{n,\pi(m)})\right)^2 \tag{61}$$

completing the proof of proposition 2.2.

***Proposition** 2.3:* $X_{1,m} = \cdots = X_{n'-1,m} = X_{n'+1,\pi(m)} = \cdots = X_{n,\pi(m)} = 0$ implies that $\det(C^T C) \geq 1$ unless $X_{n',\pi(m)} = 0$.

Proof: For a given value of n', define the function $f(.)$ as:

$$f(k) = (1 - n' - k) \mod n + 1 \tag{62}$$

Construct the $n \times n$ matrix C' given at the top of the page. For $X_{1,m} = \cdots = X_{n'-1,m} = X_{n'+1,\pi(m)} = \cdots = X_{n,\pi(m)} = 0$, C' is a lower triangular matrix. From eq. (57):

$$\det\left(C^T C\right) \geq \left(\det\left(C'\right)\right)^2 = \left(\mathrm{N}_{\mathbb{K}/\mathbb{Q}}(X_{n',\pi(m)})\right)^2 \tag{63}$$

completing the proof of proposition 2.3.

***Proposition** 2.4:* $\det(C^T C) \geq 1$ unless $C = \mathbf{0}_{nM \times n}$.

Proof: Suppose that $X_{1,m} = \cdots = X_{n-1,m}$ and $\det(C^T C) < 1$. This implies that $X_{n,m} = X_{n,\pi(m)} = 0$ from proposition 2.1 and proposition 2.2 respectively. Now, $X_{1,m} = \cdots = X_{n-2,m} = 0$ and $X_{n,\pi(m)} = 0$ imply that $X_{2,\pi(m)} = \cdots = X_{n-1,\pi(m)} = 0$ from the recursive application of proposition 2.3 for $n' = n-1, \ldots, 2$. Finally, from proposition 2.1, $X_{2,\pi(m)} = \cdots = X_{n-1,\pi(m)} = 0$ implies that $X_{1,\pi(m)} = 0$.

Therefore, $X_{1,m} = \cdots = X_{n-1,m} = 0$ and $\det(C^T C) < 1$ imply that the m-th and $\pi(m)$-th components of $X_1, \ldots, X_n$ are all equal to zero. Now, $X_{1,\pi(m)} = \cdots = X_{n-1,\pi(m)} = 0$ and $\det(C^T C) < 1$ imply that $\pi(m)$-th and $\pi^2(m)$-th components of $X_1, \ldots, X_n$ are all equal to zero. Starting the same operation again we conclude that $\det(C^T C) < 1$ only when $X_1 = \cdots = X_n = \mathbf{0}_M$ since π defines a bijection over the set $S = \{1, \ldots, M\}$ and consequently $\{\pi^1(m), \ldots, \pi^{M-1}(m)\}$ spans the entire set S for any value of $m \in S$. This proves that when $M > n^2$, $\det(C^T C) \geq 1$ for all values of M'.

APPENDIX III
INFORMATION LOSSES OF CONSTRUCTION 1

Consider the codeword C given in eq. (25). Concatenating the columns of C vertically one after the other, we obtain:

$$\mathrm{vec}(C) = \Phi \left[a_1^T \ \cdots \ a_{n^2}^T\right]^T \tag{64}$$

where Φ is a $n^2M \times n^2M$ matrix. In what follows, we will show that the matrix Φ is unitary. Φ can be written as:

$$\Phi = \left[\Phi_1^T \ \cdots \ \Phi_n^T\right]^T \tag{65}$$

Following from the layered structure of the codewords, the constituent matrices of dimensions $nM \times n^2M$ take the following form ($k = 1, \ldots, n$):

$$\Phi_k = \Psi_k \mathbf{M} \mathbf{P} \Upsilon_n^{k-1} \tag{66}$$

where $\Upsilon_n = \Omega_n^T \otimes I_{nM}$ and Ω_n is the $n \times n$ permutation matrix given by:

$$\Omega_n = \begin{bmatrix} \mathbf{0}_{1\times n} & 1 \\ I_{n-1} & \mathbf{0}_{n\times 1} \end{bmatrix} \tag{67}$$

In this case, the permutation matrix in eq. (16) can be written as Ω_M. The $nM \times nM$ matrix Ψ_k is given by:

$$\Psi_k = \mathrm{diag}(\underbrace{I_M \ \cdots \ I_M}_{k \text{ times}} \ \underbrace{\Omega_M \ \cdots \ \Omega_M}_{n-k \text{ times}}) \tag{68}$$

$\mathbf{M}$ is a $nM \times n^2M$ matrix related to the rotation matrix $\mathcal{M}$ given in eq. (25) by:

$$\mathbf{M} = \mathfrak{M} \otimes I_M = \mathrm{diag}\left(\mathcal{M}_1 \ \cdots \ \mathcal{M}_n\right) \otimes I_M \tag{69}$$

where $\mathcal{M}_i$ is the i-th row of $\mathcal{M}$. Finally, the matrix $\mathbf{P}$ is given by:

$$\mathbf{P} = P \otimes I_{nM} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 0 \end{bmatrix} \otimes I_{nM} \tag{70}$$

For example, with $n = 3$ transmit antennas:

$$\Phi_1 = \begin{bmatrix} \mathcal{M}_1 \otimes I_M & \mathbf{0}_{M\times Mn} & \mathbf{0}_{M\times Mn} \\ \mathbf{0}_{M\times Mn} & \mathbf{0}_{M\times Mn} & \mathcal{M}_2 \otimes \Omega_M \\ \mathbf{0}_{M\times Mn} & \mathcal{M}_3 \otimes \Omega_M & \mathbf{0}_{M\times Mn} \end{bmatrix}$$
$$\Phi_2 = \begin{bmatrix} \mathbf{0}_{M\times Mn} & \mathcal{M}_1 \otimes I_M & \mathbf{0}_{M\times Mn} \\ \mathcal{M}_2 \otimes I_M & \mathbf{0}_{M\times Mn} & \mathbf{0}_{M\times Mn} \\ \mathbf{0}_{M\times Mn} & \mathbf{0}_{M\times Mn} & \mathcal{M}_3 \otimes \Omega_M \end{bmatrix}$$
$$\Phi_3 = \begin{bmatrix} \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & \mathcal{M}_1 \\ \mathbf{0}_{1\times n} & \mathcal{M}_2 & \mathbf{0}_{1\times n} \\ \mathcal{M}_3 & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} \end{bmatrix} \otimes I_M$$

From eq. (66), we have $\Phi_k\Phi_k^T = I_{nM}$ since the matrices Ψ_k, $\mathbf{M}$, $\mathbf{P}$ and Υ_n are all unitary (the matrix $\mathbf{M}$ is unitary since the rotation matrix $\mathcal{M}$ is unitary by construction). Now, for any integers k and k', we have:

$$\begin{aligned}\Phi_k\Phi_{k'}^T &= \Psi_k\mathbf{M}\mathbf{P}\Upsilon_n^{k-1}(\Upsilon_n^{k'-1})^T\mathbf{P}^T\mathbf{M}^T\Psi_{k'}^T \\ &= \Psi_k\mathbf{M}\left(P\otimes I_{nM}\right)\left(\Omega_n^{1-k}\otimes I_{nM}\right) \\ &\quad \left(\Omega_n^{k'-1}\otimes I_{nM}\right)\left(P^T\otimes I_{nM}\right)\mathbf{M}^T\Psi_{k'}^T \\ &= \Psi_k\mathbf{M}\left(P\Omega_n^{k'-k}P^T\otimes I_{nM}\right)\mathbf{M}^T\Psi_{k'}^T \\ &= \Psi_k\left(\mathfrak{M}\left(P\Omega_n^{k'-k}P^T\otimes I_n\right)\mathfrak{M}^T\otimes I_M\right)\Psi_{k'}^T \end{aligned} \quad (71)$$

If a vector is multiplied by Ω_n^k, then its i-th component is permuted to the $\pi_n^k(i)$-th position where:

$$\pi_n^k(i) = (i+k-1) \mod n+1 \quad (72)$$

In the same way, multiplying by P (or P^{-1}) is equivalent to the following permutation:

$$p_n(i) = (1-i) \mod n+1 \quad (73)$$

Therefore, multiplying by the matrix $P\Omega_n^{k'-k}P^T$ is equivalent to applying the permutation:

$$f_n(i) = p_n(\pi_n^{k'-k}(p_n(i))) = \pi_n^{k-k'}(i) \quad (74)$$

Given that the matrix $\mathfrak{M}$ is block-diagonal, then the matrix $\mathfrak{M}\left(P\Omega_n^{k'-k}P^T\otimes I_n\right)\mathfrak{M}^T$ in eq. (71) is a $n\times n$ matrix having the property that each one of its rows can have at most one nonzero value. From eq. (74), the indices of the elements of this matrix that can possibly have nonzero values are equal to $(i, f_n^{-1}(i))$ for $i=1,\ldots,n$. Their values are given by:

$$\mathcal{M}_i\mathcal{M}_{f_n^{-1}(i)}^T = \mathcal{M}_i\mathcal{M}_{\pi_n^{k'-k}(i)}^T = \delta(k'-k) \quad ; \quad i=1,\ldots,n \quad (75)$$

The last equality follows from the fact that the fully diverse rotation matrix $\mathcal{M}$ is constructed to be unitary [20].

Finally, from eq. (71) it follows that:

$$\Phi_k\Phi_{k'}^T = I_{nM}\delta(k-k') \quad (76)$$

showing that the matrix Φ given in eq. (65) is unitary.

As in [30], we write the input-output baseband relation in eq. (12) as:

$$\begin{aligned}\text{vec}(Y) &= \sqrt{\frac{\rho}{n}}\left(I_T\otimes R\right)\text{vec}(C)+\text{vec}(N) \quad (77)\\ &= \sqrt{\frac{\rho}{n}}\underbrace{\left(I_T\otimes R\right)}_{R'}\Phi\left[a_1^T \ \cdots \ a_{n^2}^T\right]^T+\text{vec}(N) \quad (78)\end{aligned}$$

where ρ is the SNR and $T=n$ is the temporal extension of the ST code.

From eq. (78), the capacity of equivalent (coded) channel is given by:

$$C^{(c)}(\rho,n,r) = \frac{1}{n}E_R\log_2\left(\det\left(I_{nr}+\frac{\rho}{n}(R'\Phi)(R'\Phi)^T\right)\right) \quad (79)$$

where r is the number of rows of R and the factor $\frac{1}{n}$ follows from the number of channel uses.

Since Φ is unitary and R' is block diagonal, then eq. (79) can be written as:

$$\begin{aligned}C^{(c)}(\rho,n,r) &= \frac{1}{n}E_R\log_2\left(\left(\det\left(I_r+\frac{\rho}{n}RR^T\right)\right)^n\right) \quad (80)\\ &= E_R\log_2\left(\det\left(I_r+\frac{\rho}{n}RR^T\right)\right) \quad (81)\\ &= C(\rho,n,r) \quad (82)\end{aligned}$$

where $C(\rho,n,r)$ is the capacity of the uncoded MIM0 channel (R). This proves that the code given in eq. (25) is information lossless for all number of transmit antennas.

APPENDIX IV
CHOICE OF Ω FOR CONSTRUCTION 2

We write the $M\times M$ matrix Ω given in eq. (27) as:

$$\Omega(\theta') = I_{M/2}\otimes\Omega'(\theta') \quad ; \quad \Omega'(\theta') = \begin{bmatrix}\cos(\theta') & \sin(\theta') \\ -\sin(\theta') & \cos(\theta')\end{bmatrix} \quad (83)$$

Suppose that θ' is chosen such that $\sigma(\Omega'(\theta')) = \Omega'(\theta')$. This choice will be discussed at the end of this section. As in Appendix I, designate by $\Delta C(X_1,\ldots,X_n)$ the difference between two codewords that is associated with the vectors $X_1,\ldots,X_n$ belonging to the set $\mathcal{A}$ given in eq. (34). The linear dependence between the columns of ΔC when this latter is rank-deficient results in eq. (36) since $\Omega(\theta')$ remains invariant when applying the embeddings of $\mathbb{K}$. Equation (36) can be written in a more convenient way as:

$$\mathcal{M}(\theta')X_i = \left(\sum_{j=1}^{n-1}\lambda_j\Omega^j(\theta')\right)X_i = X_i \quad ; \quad i=1,\ldots,n \quad (84)$$

where $\lambda_1,\ldots,\lambda_{n-1}$ are rational numbers.

$\Omega'(\theta')$ is a rotation matrix verifying $(\Omega'(\theta'))^j = \Omega'(j\theta')$. This implies that $\Omega^j(\theta') = \Omega(j\theta')$ since $\Omega(\theta')$ is a block-diagonal matrix. Therefore, the matrix $\mathcal{M}(\theta')$ in eq. (84) takes the following form:

$$\mathcal{M}(\theta') = \begin{bmatrix} x(\theta') & y(\theta') & 0 & \cdots & 0 \\ -y(\theta') & x(\theta') & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & x(\theta') & y(\theta') \\ 0 & \cdots & 0 & -y(\theta') & x(\theta') \end{bmatrix} \quad (85)$$

where:

$$x(\theta') = \sum_{j=1}^{n-1}\lambda_j\cos(j\theta') \quad (86)$$

$$y(\theta') = \sum_{j=1}^{n-1}\lambda_j\sin(j\theta') \quad (87)$$

From eq. (84), X_i is an eigenvector of $\mathcal{M}(\theta')$ for $i=1,\ldots,n$. Moreover, since we are limited to real-valued constellations, then X_i can only be associated with the real eigenvalues of $\mathcal{M}(\theta')$.

Solving the characteristic equation of $\mathcal{M}(\theta')$ results in:

$$\left((x(\theta')-\mu)^2+(y(\theta'))^2\right)^{\frac{M}{2}} = 0 \quad (88)$$

where μ stands for the eigenvalue of $\mathcal{M}(\theta')$. μ is real only when $y(\theta') = 0$. In this case, X_i is the eigenvector associated with the eigenvalue $\mu = x(\theta')$. The idea is to choose θ' such that $x(\theta') = 0$ whenever $y(\theta') = 0$. In this case, if $y(\theta') \neq 0$, X_i is complex-valued and hence it does not belong to $\mathcal{A}$. When $y(\theta') = 0$, then $\mu = x(\theta') = 0$ and X_i is the all-zero vector for $i = 1, \dots, n$.

From eq. (87), this can happen when $\{\sin(\theta'), \dots, \sin((n-1)\theta')\}$ forms a $\mathbb{Z}$-basis of degree $n-1$. In this case, $y(\theta') = 0$ implies that $\lambda_1 = \cdots = \lambda_{n-1} = 0$ which implies that $x(\theta') = 0$ from eq. (86). As a special case, we choose θ' such that $\{1, \sin(\theta'), \dots, \sin((n-1)\theta')\}$ forms an n-dimensional $\mathbb{Z}$-basis. Let $\theta' = \frac{2\pi}{N'}$. The set $\{1, \exp(i\theta'), \dots, \exp((N'-1)i\theta')\}$ forms a basis of degree $\varphi(N')$ over $\mathbb{Z}$ and consequently the sets $\{1, \cos(\theta'), \dots, \cos((N'-1)\theta')\}$ and $\{1, \sin(\theta'), \dots, \sin((N'-1)\theta')\}$ form two basis of degree $\frac{\varphi(N')}{2}$ over $\mathbb{Z}$. Therefore, the desired property is obtained when $\frac{\varphi(N')}{2} \geq n$ implying that $\varphi(N') \geq 2n$.

Finally, $\mathbb{K}$ is chosen to be the real subfield of the cyclotomic field associated with the N-th root of unity ($\mathbb{K} = \mathbb{Q}(2\cos(\frac{2\pi}{N}))$ and $\varphi(N) = 2n$). Therefore, $\sigma(\Omega(\theta')) = \Omega(\theta')$ when N and N' are relatively prime.

Appendix V
Diversity Order of construction 2 with 2-PPM

For 2-PPM constellations, the elements of the set $\mathcal{A}$ given in eq. (31) can be expressed as $[a \ -a]^T$ where $a \in \mathbb{K}$. In other words, for any element $A \in \mathcal{A}$, we have $\Omega A = -A$ where Ω has the structure given in eq. (28). Therefore, the difference between two codewords takes the following form:

$$\Delta C(X_1, \dots, X_n) = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ -x_1 & -x_2 & \cdots & -x_n \\ -\sigma(x_n) & \sigma(x_1) & \ddots & \vdots \\ -\sigma(x_n) & -\sigma(x_1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \sigma^{n-2}(x_2) \\ \vdots & \ddots & \ddots & -\sigma^{n-2}(x_2) \\ -\sigma^{n-1}(x_2) & \cdots & -\sigma^{n-1}(x_n) & \sigma^{n-1}(x_1) \\ -\sigma^{n-1}(x_2) & \cdots & -\sigma^{n-1}(x_n) & -\sigma^{n-1}(x_1) \end{bmatrix} \tag{89}$$

where $X_i \in \mathcal{A}$ is written as $X_i = [x_i \ -x_i]^T$ for $i = 1, \dots, n$. When ΔC is rank deficient, consider the relation given in eq. (35) that follows from the $\mathbb{K}$-linear dependence between the columns of ΔC. The dependence between the columns of the last two rows of eq. (89) implies that:

$$\begin{cases} x_1 = -\sum_{j=1}^{n-1} \sigma(k_j) x_{\pi_n^1(j)} \\ x_1 = \sum_{j=1}^{n-1} \sigma(k_j) x_{\pi_n^1(j)} \end{cases} \tag{90}$$

where π_n^k is the permutation given in eq. (72). Equation (90) implies that the linear dependence between the columns of ΔC is possible only when $x_1 = 0$. In a more general way, the linear dependence between the columns of the $(2(n-i)+1)$-th row of eq. (89) can be written as:

$$x_i = -\sum_{j=1}^{n-i} \sigma^i(k_j) x_{\pi_n^i(j)} + \sum_{j=n-i+1}^{n-1} \sigma^i(k_j) x_{\pi_n^i(j)} \tag{91}$$

$$= -\sum_{j=1}^{n-i} \sigma^i(k_j) x_{\pi_n^i(j)} + \sum_{j=1}^{i-1} \sigma^i \left(k_{\pi_n^{-i}(j)}\right) x_j \tag{92}$$

When $x_1 = \cdots = x_{i-1} = 0$, this implies that:

$$x_i = -\sum_{j=1}^{n-i} \sigma^i(k_j) x_{\pi_n^i(j)} \tag{93}$$

In the same way, the linear dependence between the columns of the $(2(n-i)+2)$-th row of eq. (89) when $x_1 = \cdots = x_{i-1} = 0$ can be written as:

$$x_i = \sum_{j=1}^{n-i} \sigma^i(k_j) x_{\pi_n^i(j)} \tag{94}$$

Equations (93) and (94) imply that $x_i = 0$. Applying eq. (93) and eq. (94) recursively for $i = 2, \dots, n$ along with eq. (90) (for x_1) imply that $x_1 = \cdots = x_n = 0$. From eq. (89), this implies that the only rank-deficient matrix ΔC is the all-zero matrix. This shows that the choice of Ω given in eq. (28) results in fully diverse ST codes with 2-PPM constellations for any number of transmit antennas.

Appendix VI
Coding Gain of construction 2 with 2-PPM

In a way similar to eq. (57), $\det(\Delta C^T \Delta C)$ verifies the following relation:

$$\det(\Delta C^T \Delta C) \geq \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} \left(\det\left(\left[(\Delta C_{i_1}^{(0)})^T \ \cdots \ (\Delta C_{i_n}^{(n-1)})^T \right]^T \right) \right)^2 \tag{95}$$

where $\Delta C_j^{(i)}$ is the j-th row of the $2 \times n$ matrix $\Delta C^{(i)}$ composed of the $(2i+1)$-th and the $2(i+1)$-th rows of ΔC. Following from eq. (89), $\Delta C^{(i)}$ has the following form (for $i = 0, \dots, n-1$):

$$\Delta C^{(i)} = \sigma^i \begin{bmatrix} -x_{n-i+1} & \cdots & -x_n & x_1 & \cdots & x_{n-i} \\ -x_{n-i+1} & \cdots & -x_n & -x_1 & \cdots & -x_{n-i} \end{bmatrix} \tag{96}$$

Note that $\Delta C_1^{(0)} = -\Delta C_2^{(0)} = [x_1 \ \cdots \ x_n]$. Given that the absolute value of the determinant of a matrix remains invariant if one of its rows is multiplied by -1, then $\Delta C_2^{(i)}$ can be replaced by $-\Delta C_2^{(i)}$ in eq. (95). Therefore, eq. (95) can be written as:

$$\det(\Delta C^T \Delta C) \geq 2\sum_{i_2=1}^{2} \cdots \sum_{i_n=1}^{2} \left(\det\left(\left[(\Delta C_1^{(0)})^T \ (\Delta C_{i_2}^{(1)})^T \cdots (\Delta C_{i_n}^{(n-1)})^T \right]^T \right) \right)^2$$

$$\triangleq 2 \sum_{i_2=1}^{2} \cdots \sum_{i_n=1}^{2} \left(\det\left(\mathcal{C}'(i_2, \dots, i_n) \right) \right)^2 \tag{97}$$

$$\mathcal{C}'(i_2,\ldots,i_n) = \left[\begin{array}{cccc} (\Delta C_1^{(0)})^T & (-1)^{i_2}(\Delta C_{i_2}^{(1)})^T & \cdots & (-1)^{i_n}(\Delta C_{i_n}^{(n-1)})^T \end{array}\right]^T$$
$$= \underbrace{\left[\begin{array}{ccccc} x_1 & x_2 & x_3 & \cdots & x_n \\ j_1\sigma(x_n) & \sigma(x_1) & \sigma(x_2) & \cdots & \sigma(x_{n-1}) \\ j_2\sigma^2(x_{n-1}) & j_2\sigma^2(x_n) & \sigma^2(x_1) & \cdots & \sigma^2(x_{n-2}) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ j_{n-1}\sigma^{n-1}(x_2) & j_{n-1}\sigma^{n-1}(x_3) & \cdots & j_{n-1}\sigma^{n-1}(x_n) & \sigma^{n-1}(x_1) \end{array}\right]}_{\mathcal{C}(j_1,\ldots,j_{n-1})} \tag{98}$$

where the matrix $\mathcal{C}'(i_2,\ldots,i_n)$ is given in eq. (98) at the top of the page. In the expression of this matrix, we fix $j_k = (-1)^{i_{k+1}}$ for $k = 1,\ldots,n-1$.

Therefore, eq. (97) can be written as:

$$\det(\Delta C^T\Delta C) \geq 2\sum_{j_1=\pm1}\cdots\sum_{j_{n-1}=\pm1}\underbrace{(\det(\mathcal{C}(j_1,\ldots,j_{n-1})))^2}_{\delta(j_1,\ldots,j_{n-1})} \tag{99}$$

Since j_k appears in only one row of $\mathcal{C}(j_1,\ldots,j_{n-1})$ for $k = 0,\ldots,n-1$, then the highest power of j_k that appears in the determinant of $\mathcal{C}(j_1,\ldots,j_{n-1})$ is equal to one for the different values of k. Consequently, $\delta(j_1,\ldots,j_{n-1})$ can be expressed as:

$$\delta(j_1,\ldots,j_{n-1}) = \left(\sum_{(p_1,\ldots,p_{n-1})\in\{0,1\}^{n-1}} c_{p_1,\ldots,p_{n-1}}(j_1,\ldots,j_{n-1})f_{p_1,\ldots,p_{n-1}}\right)^2 \tag{100}$$

$$= \sum_{\substack{(p_1,\ldots,p_{n-1})\in\{0,1\}^{n-1}\\(p'_1,\ldots,p'_{n-1})\in\{0,1\}^{n-1}}}\Big\{c_{p_1,\ldots,p_{n-1}}(j_1,\ldots,j_{n-1})c_{p'_1,\ldots,p'_{n-1}}(j_1,\ldots,j_{n-1})f_{p_1,\ldots,p_{n-1}}f_{p'_1,\ldots,p'_{n-1}}\Big\} \tag{101}$$

where the coefficients $c_{p_1,\ldots,p_{n-1}}(j_1,\ldots,j_{n-1})$ are given by:

$$c_{p_1,\ldots,p_{n-1}}(j_1,\ldots,j_{n-1}) = \prod_{l=1}^{n-1} j_l^{p_l} \tag{102}$$

$f_{p_1,\ldots,p_{n-1}}$ is a certain function of $x_1,\ldots,x_n$ and their conjugates (it can be equal to zero). For example, when $n = 3$:

$$\begin{cases} f_{0,0} = \mathrm{N}_{\mathbb{K}/\mathbb{Q}}(x_1) \\ f_{1,0} = -x_2\sigma(x_3)\sigma^2(x_1) \\ f_{0,1} = \mathrm{N}_{\mathbb{K}/\mathbb{Q}}(x_2) - x_1\sigma(x_2)\sigma^2(x_3) - x_3\sigma(x_1)\sigma^2(x_2) \\ f_{1,1} = \mathrm{N}_{\mathbb{K}/\mathbb{Q}}(x_3) \end{cases}$$

From the structure of the matrix given in eq. (98), the following relations hold for any value of n:

$$f_{0,\ldots,0} = \mathrm{N}_{\mathbb{K}/\mathbb{Q}}(x_1) \quad ; \quad f_{1,\ldots,1} = (-1)^{n+1}\mathrm{N}_{\mathbb{K}/\mathbb{Q}}(x_n) \tag{103}$$

Given that $p_l \in \{0,1\}$ for $l = 1,\ldots,n-1$, from eq. (102), it follows that:

$$\sum_{j_1=\pm1}\cdots\sum_{j_{n-1}=\pm1} c_{p_1,\ldots,p_{n-1}}(j_1,\ldots,j_{n-1})c_{p'_1,\ldots,p'_{n-1}}(j_1,\ldots,j_{n-1})$$
$$= \sum_{j_1=\pm1}(j_1^{p_1+p'_1}) \quad \cdots \sum_{j_{n-1}=\pm1}(j_{n-1}^{p_n+p'_n}) \tag{104}$$
$$= \begin{cases} 0\ ; & (p_1,\ldots,p_{n-1}) \neq (p'_1,\ldots,p'_{n-1}) \\ 2^{n-1}\ ; & (p_1,\ldots,p_{n-1}) = (p'_1,\ldots,p'_{n-1}) \end{cases} \tag{105}$$

Therefore, from eq. (99):

$$\det(\Delta C^T\Delta C) \geq 2^n \sum_{(p_1,\ldots,p_{n-1})\in\{0,1\}^{n-1}} f^2_{p_1,\ldots,p_{n-1}} \tag{106}$$
$$\geq 2^n\left(f^2_{0,\ldots,0} + f^2_{1,\ldots,1}\right) \tag{107}$$
$$= 2^n\left(\left(\mathrm{N}_{\mathbb{K}/\mathbb{Q}}(x_1)\right)^2 + \left(\mathrm{N}_{\mathbb{K}/\mathbb{Q}}(x_n)\right)^2\right) \tag{108}$$

Since $\mathrm{N}_{\mathbb{K}/\mathbb{Q}}(x_i) \in \mathbb{Z}$ for all values of i, this implies that $\det(\Delta C^T\Delta C) \geq 2^n$ unless when $\mathrm{N}_{\mathbb{K}/\mathbb{Q}}(x_1) = \mathrm{N}_{\mathbb{K}/\mathbb{Q}}(x_n) = 0$.

Suppose that $\mathrm{N}_{\mathbb{K}/\mathbb{Q}}(x_1) = \mathrm{N}_{\mathbb{K}/\mathbb{Q}}(x_n) = 0$ implying that $x_n = 0$. The only time when j_1 appears in eq. (98), it is multiplied by $\sigma(x_n)$ which is equal to zero. Therefore, j_1 will not appear neither in the expression of $c_{p_1,\ldots,p_{n-1}(j_1,\ldots,j_{n-1})}$ nor in that of $\delta(j_1,\ldots,j_{n-1})$. This implies that eq. (101) and eq. (102) can now be written as:

$$\delta(j_2,\ldots,j_{n-1}) = \left(\sum_{(p_2,\ldots,p_{n-1})\in\{0,1\}^{n-2}} c_{p_2,\ldots,p_{n-1}}(j_2,\ldots,j_{n-1})f_{p_2,\ldots,p_{n-1}}\right)^2 \tag{109}$$

where:

$$c_{p_2,\ldots,p_{n-1}}(j_2,\ldots,j_{n-1}) = \prod_{l=2}^{n-1} j_l^{p_l} \tag{110}$$

with $f_{1,\ldots,1} = \pm\mathrm{N}_{\mathbb{K}/\mathbb{Q}}(x_{n-1})$. As in eq. (105), this implies that:

$$\det(\Delta C^T\Delta C) \geq 2^n \sum_{(p_2,\ldots,p_{n-1})\in\{0,1\}^{n-2}} f^2_{p_2,\ldots,p_{n-1}} \tag{111}$$
$$\geq 2^n\left(\mathrm{N}_{\mathbb{K}/\mathbb{Q}}(x_{n-1})\right)^2 \tag{112}$$

implying that $\det(\Delta C^T\Delta C) \geq 2^n$ unless $x_{n-1} = 0$. Proceeding recursively in the same way, and noticing that at step k, j_k is multiplied by the conjugates of $x_n,\ldots,x_{n-k+1}$ in eq. (98), we deduce that $\det(\Delta C^T\Delta C) \geq 2^n$ unless $x_1 = \cdots = x_n = 0$.

As in [1], [5], [6], limiting the construction in an ideal whose fundamental parallelotope has a volume of 1 is equivalent to multiplying the codewords by a diagonal matrix whose determinant is equal to $\frac{1}{\sqrt{d_{\mathbb{K}}}}$ where $d_{\mathbb{K}}$ is the absolute discriminant of $\mathbb{K}$. Therefore, the coding gain of the proposed code with 2-PPM and n transmit antennas is equal to:

$$\delta_{min} = \min_{\substack{\Delta C(X_1,\dots,X_n) \\ (X_1,\dots,X_n)\in\mathcal{A}^n\setminus\{(\mathbf{0}_{2\times1},\dots,\mathbf{0}_{2\times1})\}}} \left(\det\left(\Delta C^T \Delta C\right)\right)^{\frac{1}{n}} = 2d_{\mathbb{K}}^{-\frac{1}{n}} \tag{113}$$

On the other hand, the perfect codes achieve a coding gain of $d_{\mathbb{K}}^{-\frac{1}{n}}$ over $\mathbb{Z}$ [1] ($d_\infty = d_{\mathbb{K}}^{-\frac{1}{n}}$ in eq. (4)). From eq. (4), this implies that the coding gain is equal to $4d_{\mathbb{K}}^{-\frac{1}{n}}$ and $2d_{\mathbb{K}}^{-\frac{1}{n}}$ with PAM and M-PPM-M'-PAM constellations respectively (with $M > 1$). This shows that the proposed scheme achieves the same coding gain as the perfect codes.

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Chadi Abou-Rjeily received his B.S. degree in computer and electrical engineering in 2002 from the Lebanese University, Roumieh, Lebanon. He received his M.S. and Ph.D. degrees in electrical engineering in 2003 and 2006 respectively from the " École Nationale Supérieure des Télécommunications (ENST)", Paris, France. From September 2003 to February 2007, he was also a research fellow at the Laboratory of Electronics and Information Technology of the French Atomic Energy Commission (CEA-LETI). Currently, he is an assistant professor at the faculty of engineering and architecture of the Lebanese American University, Byblos, Lebanon. His research interests are in space-time coding and ultra wideband communications.

Jean-Claude Belfiore was born in Nice, France, in 1962. He received his Engineer degree in Electrical Engineering in 1985 from Supelec, France and his Ph.D. in 1989 from ENST, Paris, France. From 1985 to 1988 he was also with Alcatel, France. He then became assistant professor at ENST where he is now a professor in the Communications and Electronics Department. His current research interests are in lattice coding for fading channels, space-time coding, joint source-channel coding and wireless communications systems.