

Power Allocation for Quantum-Limited Multihop Free-Space Optical Communication Systems

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Abstract—In this paper, we propose two novel power allocation strategies for multihop Free-Space Optical (FSO) networks with any number of hops. These strategies correspond to the analytical solutions obtained from minimizing two different upper-bounds on the conditional error probability under the quantum-limit scenario. The performance of the first strategy is evaluated numerically and closed-form expressions of the average probability of error are derived for the second strategy. We also quantify the reduction in the fading variance and scintillation index provided by multihop FSO systems associated with the second power allocation scheme that compromises performance for simplicity. Results show the superiority of the proposed schemes compared to the equal power distribution strategy.

Index Terms—Free-space optics, FSO, cooperation, relaying.

I. INTRODUCTION

While the literature on cooperation in Radio-Frequency (RF) wireless networks is huge and dates back to more than a decade, it was only recently that cooperation techniques started attracting an increasing interest in the context of Free-Space Optical (FSO) communications [1]–[6]. Cooperative diversity is capable of leveraging the performance of FSO systems by mitigating the limiting effects of the turbulence-induced fading or scintillation. Amplify-and-forward (AF) and decode-and-forward (DF) parallel relaying with one relay were considered in [1] and [2], respectively. An optimal power allocation strategy for DF parallel relaying with any number of relays was proposed in [3] under the quantum-limit scenario. On the other hand, multihop systems that are based on serial relaying were considered in [4]–[6]. Unlike RF systems where the fading variance is constant, FSO systems are characterized by a distance-dependent fading variance which privileges numerous short hops over a single longer hop. Consequently, unlike multihop RF systems, multihop FSO systems are capable of combatting fading along with their coverage broadening capabilities. High performance gains have been reported especially in [4] where an aggregated channel model, taking both path-loss and fading into account, was considered.

The multihop FSO systems studied in [4]–[6] are all based on a simple power allocation strategy (PAS) that consists of evenly distributing the transmit power among the different hops. While this strategy is simple and suitable for systems where the channel state information (CSI) is not available at the communicating nodes, higher performance gains can be realized by adapting the power distribution to the strengths of the hops. In this paper, we propose and analyze two novel

PAS's in the context where the CSI is known. While the first PAS is optimal in the sense of minimizing a tight upper-bound on the conditional error probability, the second solution is appealing because of its remarked simplicity. A numerical analysis of the first PAS and an analytical evaluation of the second PAS showed that significant improvements can be achieved compared to the even power distribution strategy.

II. SYSTEM MODEL

Consider a multihop FSO system composed of N_h hops between the source node (S) and the destination node (D). In this case, the data is relayed serially by $N_h - 1$ nodes or relays denoted by R_1, \dots, R_{N_h-1} . The distance of the n -th hop is denoted by d_n and the fraction of the power transmitted by the $(n - 1)$ -th node along the n -th hop is denoted by P_n for $n = 1, \dots, N_h$ where, by abuse of notation, S and D are denoted by R_0 and R_{N_h} , respectively. For the multihop system to transmit the same average power as single-hop systems, the following relation must be satisfied: $\sum_{n=1}^{N_h} P_n = 1$.

We consider Q -ary Pulse Position Modulation (PPM) with intensity modulation and direct detection (IM/DD). In this work, we will assume that the FSO system is operating under the quantum-limit scenario where the background radiation and dark currents are neglected. This approach results in simple closed-form PAS's that can be easily implemented in realistic FSO systems. The cooperation strategy is as follows. If the n -th relay detects a nonzero photoelectron count in one PPM slot, then it decides in favor of this slot and forwards the corresponding PPM symbol along the $(n + 1)$ -th hop. In the absence of background radiation, this nonzero count ensures that the decision made at this relay is correct. On the other hand, if all Q slots are empty, then the relay R_n backs off in order to avoid confusing the relay R_{n+1} (and eventually D) by forwarding a wrong estimate of the symbol. Note that the same cooperation strategy can be applied with On-Off Keying (OOK) where the "On" and "Off" signals are forwarded by R_n in the case of nonzero and zero counts, respectively.

The average number of photoelectrons generated by the incident light signal in a PPM slot is given by [2]:

$$\lambda_s = \eta \frac{P_r T_s / Q}{hf} = \eta \frac{E_s}{hf} \quad (1)$$

where η is the detector's quantum efficiency assumed to be equal to 0.5, $h = 6.6 \times 10^{-34}$ is Planck's constant and f is the optical center frequency taken to be 1.94×10^{14} Hz (corresponding to a wavelength of 1550 nm). T_s stands for the symbol duration while P_r stands for the optical power that falls on the receiver corresponding to the optical signal transmitted

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over the direct link. Finally, $E_s = P_r T_s / Q$ corresponds to the received optical energy per slot along the direct link. For OOK, λ_s stands for the average number of photoelectrons per symbol which can be obtained by setting $Q = 1$ in (1).

Consider the transmission over the n -th hop R_{n-1} - R_n . The decision at the n -th node will be based on the Q -dimensional vector $\mathbf{Z}_n = [Z_{n,1}, \dots, Z_{n,Q}]$ where $Z_{n,q}$ corresponds to the number of photoelectrons detected by R_n in the q -th PPM slot. In the case where the previous node R_{n-1} backed off, \mathbf{Z}_n will be equal to the all-zero vector. On the other hand, if the PPM symbol $s \in \{1, \dots, Q\}$ was transmitted by R_{n-1} , then $Z_{n,q} = 0$ for $q \neq s$ since in the absence of background radiation the only source of photoelectrons is the light signal itself. In this case, $Z_{n,s}$ can be modeled as a Poisson random variable (r.v.) whose parameter is given by:

$$E[Z_{n,s}] = P_n \beta_n a_n^2 \lambda_s \triangleq P_n k_n \quad (2)$$

where a_n stands for the path gain of the n -th hop. In this work, we adopt the lognormal turbulence-induced fading channel model [4]. In this case, the probability density function (pdf) of the path gain a_n (where $a_n \geq 0$) is given by:

$$f(a_n) = \frac{1}{\sqrt{2\pi}\sigma_n a_n} \exp\left(-\frac{(\ln(a_n) - \mu_n)^2}{2\sigma_n^2}\right) \quad (3)$$

where the parameters μ_n and σ_n satisfy the relation $\mu_n = -\sigma_n^2$ so that the mean path intensity is unity: $E[I_n] = E[a_n^2] = 1$. We adopt the same channel model as in [4] where the distance-dependent log-amplitude variance is given by:

$$\sigma_n^2(d_n) = 0.124k^{7/6} C_n^2 d_n^{11/6} \quad (4)$$

where k is the wave number and C_n^2 denotes the refractive index structure constant [4]. In (2), β_n is a gain factor associated with the n -th hop and resulting from the fact that this hop is shorter than the direct link S-D [4]:

$$\beta_n = \left(\frac{d}{d_n}\right)^2 e^{-\sigma(d_n-d)} \quad (5)$$

where σ is the attenuation coefficient and $d = \sum_{n=1}^{N_h} d_n$ is the distance between S and D.

III. POWER ALLOCATION AND PERFORMANCE ANALYSIS

A. Conditional Error Probability

The channel state is defined by $K \triangleq [k_1, \dots, k_{N_h}]$. For PPM, if one relay among R_1, \dots, R_{N_h-1} backs off, no signal will reach D and the random tie breaking at this node results in an erroneous decision with probability $\frac{Q-1}{Q}$. On the other hand, a correct decision can be ensured at D only when nonzero counts are observed along all hops. Given that the probability of nonzero count along the n -th hop is $1 - e^{-k_n P_n}$, the conditional error probability can be written as:

$$P_{e|K} = \frac{Q-1}{Q} \left[1 - \prod_{n=1}^{N_h} (1 - e^{-k_n P_n}) \right] \quad (6)$$

For the multihop systems considered in [4]–[6], the power is evenly distributed among the N_h hops resulting in $P_n = 1/N_h$ for all values of n . This power allocation strategy will

be referred to as PAS-0 in what follows. We next propose two novel strategies under the scenario where the vector K is known by the different transmitting nodes in the network. Note that for OOK, $P_{e|K}$ can be obtained by replacing the multiplying factor $\frac{Q-1}{Q}$ by $\frac{1}{2}$ in (6) implying that the proposed strategies can be applied with OOK as well.

B. First Power Allocation Strategy: PAS-1

For large values of λ_s , (6) can be upper-bounded by:

$$P_{e|K} \leq \frac{Q-1}{Q} \sum_{n=1}^{N_h} e^{-k_n P_n} \quad (7)$$

The first power allocation strategy, denoted by PAS-1 in what follows, will be based on minimizing the asymptotic expression in (7) subject to the constraints: $\sum_{n=1}^{N_h} P_n = 1$ and $P_n \geq 0$ for $n = 1, \dots, N_h$.

In the appendix, we perform this constrained minimization by applying the method of Lagrange multipliers with the solution satisfying the Karush-Kuhn-Tucker (KKT) conditions for ensuring non-negative powers. The solution is given by:

$$P_n = \frac{1}{k_n} \frac{1 + \sum_{i=1}^{N_h} \frac{1}{k_i} \ln\left(\frac{k_n}{k_i}\right)}{\sum_{i=1}^{N_h} \frac{1}{k_i}} ; \quad n = 1, \dots, N_h \quad (8)$$

C. Second Power Allocation Strategy: PAS-2

For the second power allocation strategy, denoted by PAS-2, the expression in (7) will be further upper-bounded by:

$$P_{e|K} \leq \frac{Q-1}{Q} N_h e^{-\min\{k_1 P_1, \dots, k_{N_h} P_{N_h}\}} \quad (9)$$

where minimizing the above expression is equivalent to maximizing $\min\{k_1 P_1, \dots, k_{N_h} P_{N_h}\}$.

Assume that the functions $k_1 P_1, \dots, k_{N_h} P_{N_h}$ intersect at a certain point. In this case, $\min\{k_1 P_1, \dots, k_{N_h} P_{N_h}\}$ will be maximized at this point of intersection. In fact, for any other point, one of the functions $k_1 P_1, \dots, k_{N_h} P_{N_h}$ will be smaller than the others and $\min\{k_1 P_1, \dots, k_{N_h} P_{N_h}\}$ will decrease. Consequently, (9) is minimized when $k_1 P_1 = \dots = k_{N_h} P_{N_h}$. Solving this system of equations results in:

$$P_n = \frac{1}{k_n} \frac{1}{\sum_{i=1}^{N_h} \frac{1}{k_i}} ; \quad n = 1, \dots, N_h \quad (10)$$

D. Average Error Probability

For PAS-0, integrating (6) with respect to the lognormal distributions of the path gains results in:

$$P_e = \frac{Q-1}{Q} \left[1 - \prod_{n=1}^{N_h} \left[1 - \text{Fr}\left(\frac{\beta_n \lambda_s}{N_h}, 0; \sigma_n\right) \right] \right] \quad (11)$$

where $\text{Fr}(a, 0; b)$ is the lognormal density frustration function $\text{Fr}(a, 0, b) = \int_0^{+\infty} \frac{1}{\sqrt{2\pi b x}} \exp(-ax^2) \exp\left[-\frac{(\ln(x)+b^2)^2}{2b^2}\right] dx$ [7].

We next evaluate the average error performance of PAS-2. From (10), we observe that $k_n P_n = \left(\sum_{i=1}^{N_h} \frac{1}{k_i}\right)^{-1} \triangleq k_{eq}$ for all values of n . Consequently, (6) can be written as:

$$P_{e|K} = \frac{Q-1}{Q} \left[1 - (1 - e^{-k_{eq}})^{N_h} \right] \quad (12)$$

Since the path gain a_n is a lognormal r.v. with parameters μ_n and σ_n , then $k_n^{-1} = (\beta_n a_n^2 \lambda_s)^{-1}$ is also a lognormal r.v. with parameters $-\ln(\beta_n) - \ln(\lambda_s) - 2\mu_n$ and $2\sigma_n$. Consequently, the term $\sum_{n=1}^{N_h} \frac{1}{k_n}$ corresponds to the summation of N_h lognormal r.v.s and thus can be accurately approximated by a lognormal distribution [8]. As a conclusion, k_{eq}^{-1} (as well as k_{eq}) can be approximated by a lognormal distribution.

A simple method for determining the parameters μ_{eq} and σ_{eq} of the equivalent lognormal r.v. k_{eq} is by applying the Wilkinson's method [8]. In this case, matching the first moments of k_{eq}^{-1} and $\sum_{n=1}^{N_h} \frac{1}{k_n}$ results in:

$$M_1 = e^{-\mu_{eq} + \frac{1}{2}\sigma_{eq}^2} = \sum_{n=1}^{N_h} e^{-\ln(\beta_n) - \ln(\lambda_s) - 2\mu_n + 2\sigma_n^2} \quad (13)$$

$$= \frac{1}{\lambda_s} \sum_{n=1}^{N_h} \frac{1}{\beta_n} e^{4\sigma_n^2} \quad (14)$$

where (14) follows from $\mu_n = -\sigma_n^2$.

In the same way, matching the second moments results in:

$$M_2 = e^{-2\mu_{eq} + 2\sigma_{eq}^2} \quad (15)$$

$$= \frac{1}{\lambda_s^2} \left[\sum_{n=1}^{N_h} \frac{1}{\beta_n^2} e^{12\sigma_n^2} + 2 \sum_{\substack{n=1 \\ n' > n}}^{N_h} \frac{1}{\beta_n \beta_{n'}} e^{4\sigma_n^2 + 4\sigma_{n'}^2} \right] \quad (16)$$

Solving (13)-(16) for μ_{eq} and σ_{eq} results in:

$$\mu_{eq} = \ln \left(\frac{M_2^{1/2}}{M_1^2} \right) ; \quad \sigma_{eq}^2 = \ln \left(\frac{M_2}{M_1^2} \right) \quad (17)$$

Finally, invoking the binomial formula in (12) and integrating this equation with respect to a lognormal distribution with parameters μ_{eq} and σ_{eq} results in:

$$P_e = \frac{Q-1}{Q} \sum_{n=1}^{N_h} \binom{N_h}{n} (-1)^{n+1} \text{Fr} \left(n e^{\mu_{eq} + \frac{1}{2}\sigma_{eq}^2}, 0; \frac{\sigma_{eq}}{2} \right) \quad (18)$$

We next determine the reduction in the log-amplitude variance offered by multihop systems in the case of equidistant relays. In this case, $d_1 = \dots = d_{N_h} = \frac{d}{N_h}$ resulting in $\beta_1 = \dots = \beta_{N_h} \triangleq \beta$ and $\sigma_1 = \dots = \sigma_{N_h} \triangleq \sigma$. In this case, (14) and (16) can be written as $M_1 = \frac{N_h}{\beta \lambda_s} e^{4\sigma^2}$ and $M_2 = \frac{N_h}{\beta^2 \lambda_s^2} \left[e^{12\sigma^2} + (N_h - 1)e^{8\sigma^2} \right]$ resulting in (from (17)): $\sigma_{eq}^2 = 4\sigma^2 + \ln \left(\frac{1 + (N_h - 1)e^{-4\sigma^2}}{N_h} \right)$. Note that k_{eq} is proportional to the square of a path gain. Writing the equivalent path gain as $a_{eq} = k_{eq}^{1/2}$, then a_{eq} is a lognormal r.v. with parameters $\mu'_{eq} = \frac{\mu_{eq}}{2}$ and $\sigma'_{eq} = \frac{\sigma_{eq}}{2}$ where:

$$\sigma_{eq}'^2 = \sigma^2 + \frac{1}{4} \ln \left(\frac{1 + (N_h - 1)e^{-4\sigma^2}}{N_h} \right) \quad (19)$$

In other words, the variance of the logarithm of the path gain decreases from σ^2 in the case of single-hop systems ($N_h = 1$) to the value given in (19) for a N_h -hop system. Unfortunately, PAS-1 does not lend itself to a simple analytical evaluation since the statistical distribution of the term $k_n P_n$ in (8) can not be obtained in closed form.

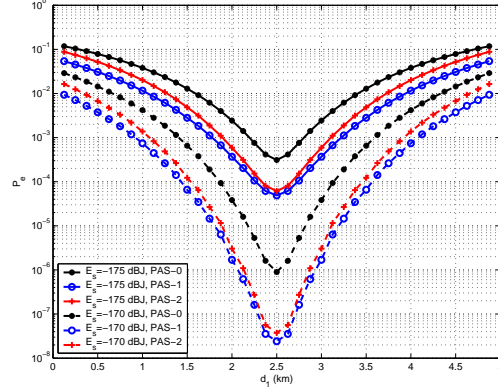


Fig. 1. Performance of a 2-hop system with 4-PPM.

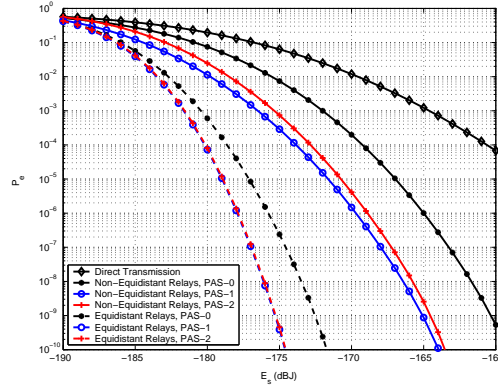


Fig. 2. Performance of a 3-hop system with 4-PPM.

IV. NUMERICAL RESULTS

We next compare the different power allocation strategies with 4-PPM. We set $C_n^2 = 1 \times 10^{-14} \text{ m}^{-2/3}$ and $\sigma = 0.43$ dB/km in (4) and (5), respectively. In all scenarios, the link range (distance between S and D) is $d = 5$ km. The numerical analysis showed that (11) and (18) can accurately predict the performance of PAS-0 and PAS-2, respectively. Results also showed that PAS-1 is extremely close to the optimal solution obtained from numerically minimizing (6). On the other hand, the bound in (7) is extremely close to the exact results for practical values of E_s exceeding -190 dBJ.

Fig. 1 shows the performance of a 2-hop system as a function of d_1 (distance of the first hop). Results show the superiority of the proposed schemes PAS-1 and PAS-2 compared to PAS-0. For all schemes, the best performance is achieved when the relay is midway between S and D. This placement avoids any amplification in the fading variance since this parameter increases with the link distance (4). In this case, for PAS-2 σ_{eq}' in (19) decreases from 0.62 to 0.24 as the number of hops N_h increases from 1 to 2.

Fig. 2 shows the performance with 3 hops. In particular, we compare the two scenarios where the consecutive nodes are equidistant and not equidistant (along the S-D link). In the latter case, we set $d_1 = d_2 = 1$ km and $d_3 = 3$ km. The results show that the first scenario results in higher performance levels. Moreover, the performance levels achieved by the optimal strategy PAS-1 and the simplified strategy PAS-

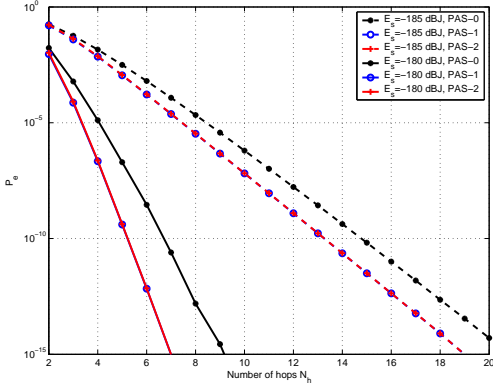


Fig. 3. Impact of the number of hops with 4-PPM.

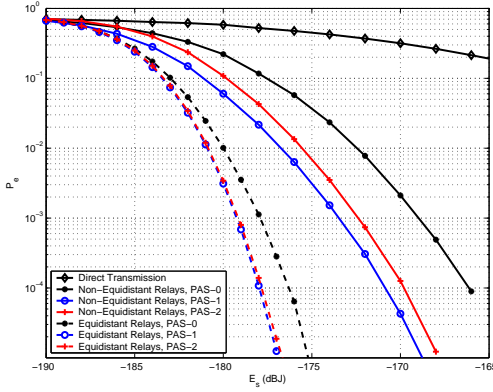


Fig. 4. Impact of background radiation on a 3-hop system with 4-PPM.

2 are comparable especially in the case of equidistant relays. In the case of non-equidistant relays, PAS-1 outperforms PAS-2 by about 1 dB. At $E_s = -170$ dB, P_e drops from 2×10^{-4} to about 10^{-6} by implementing PAS-1 instead of PAS-0. This corresponds to a power of 40 nW at a data rate of 1 Gbits/s.

Fig. 3 shows the variation of P_e as a function of N_h in the case of equidistant relays. For all values of N_h , PAS-1 and PAS-2 manifest comparable performance levels that are higher than those achieved by PAS-0.

Fig. 4 shows the impact of background radiation on the performance of a 3-hop system. 4-PPM is used and the energy of background noise per PPM slot is fixed to $E_b = -185$ dBJ. Results show that the proposed PAS's that are designed for the quantum-limit scenario achieve high performance levels in the presence of background noise as well. The enhancements over the even power distribution strategy are evident for equidistant and non-equidistant relay positions.

V. CONCLUSION

The level of fading along a FSO link depends on its distance thus rendering multihop relaying a simple and efficient fading mitigation technique for FSO systems. The achievable performance levels can be further boosted by adapting the transmitted powers to the relative strengths of the links in the case where the full CSI is known. These adaptive solutions can profit from the large coherence time of FSO systems where the estimates of the underlying channels, and hence

the transmitted powers, do not need to be updated very often. Appropriate channel estimation techniques and PAS's with partial CSI constitute the subject of future work.

APPENDIX

From (7), minimizing $P_{e|K}$ is equivalent to minimizing $\sum e^{-k_n P_n}$. In order to take into consideration the equality constraint $\sum_{n=1}^{N_h} P_n = 1$ and the inequality constraints $-P_n \leq 0$ for $n = 1, \dots, N_h$, we construct the Lagrangian function:

$$\Lambda = \sum_{n=1}^{N_h} e^{-k_n P_n} + \lambda \left(\sum_{n=1}^{N_h} P_n - 1 \right) - \sum_{n=1}^{N_h} \mu_n P_n \quad (20)$$

Λ must satisfy the following $N_h + 1$ equations:

$$\frac{\partial \Lambda}{\partial P_n} = -k_n e^{-k_n P_n} + \lambda - \mu_n = 0 ; n = 1, \dots, N_h \quad (21)$$

$$\frac{\partial \Lambda}{\partial \lambda} = \sum_{n=1}^{N_h} P_n - 1 = 0 \quad (22)$$

The KKT conditions are summarized in a set of N_h equalities and N_h inequalities as follows:

$$\mu_n P_n = 0 ; n = 1, \dots, N_h \quad (23)$$

$$\mu_n \geq 0 ; n = 1, \dots, N_h \quad (24)$$

For the multihop system, if $P_n = 0$ for a certain value of n , then relays R_n, \dots, R_{N_h-1} will stop their retransmissions and no signal will be relayed to D. Consequently, $P_n > 0$ for $n = 1, \dots, N_h$ resulting in $\mu_1 = \dots = \mu_{N_h} = 0$ from (23). Consequently, the N_h inequalities in (24) are satisfied. Replacing $\mu_n = 0$ in (21) results in $\lambda = k_n e^{-k_n P_n}$ which can be written as:

$$P_n = \frac{1}{k_n} \ln \left(\frac{k_n}{\lambda} \right) ; n = 1, \dots, N_h \quad (25)$$

Replacing (25) in (22) and solving for λ results in:

$$\ln(\lambda) = \frac{\sum_{n=1}^{N_h} \frac{1}{k_n} \ln(k_n) - 1}{\sum_{n=1}^{N_h} \frac{1}{k_n}} \quad (26)$$

Finally, replacing (26) in (25) results in (8).

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