Cooperative FSO Systems: Performance Analysis and Optimal Power Allocation

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***Abstract*— In this paper, we investigate the cooperative diversity technique as a candidate solution for combating turbulence-induced fading over Free-Space Optical (FSO) links. In particular, we propose a novel cooperation strategy that is suitable for quantum-limited FSO systems with any number of relays and we derive closed-form expressions for the error performance of this strategy. In scenarios where the Channel-State-Information (CSI) is available at the different nodes, we propose an optimal power allocation strategy that satisfies the Karush-Kuhn-Tucker (KKT) conditions and that further boosts the performance of FSO networks. It turned out that this closed-form optimal solution corresponds to transmitting the entire optical power along the "strongest link" between the source and the destination nodes. A simple procedure is proposed for selecting this link and for distributing the power among its different hops.**

***Index Terms*— Free-space optics, spatial diversity, cooperative diversity, atmospheric turbulence, power allocation.**

I. Introduction

Recently, Free-Space Optical (FSO) communications attracted significant attention as a promising solution for the "last mile" problem [1]. A major impairment that severely degrades the link performance is fading (or scintillation) that results from the variations of the index of refraction due to inhomogeneities in temperature and pressure changes [2]. In order to combat fading, the Multiple-Input-Multiple-Output (MIMO) techniques, that were extensively studied in the context of RF communications, were recently extended and tailored to FSO systems [3]–[5]. In this context, it is well known that MIMO systems achieve the highest performance gains in the case of spatially uncorrelated channels. For RF systems, the assumption of uncorrelated channels is often justified since the wide beamwidth of the antennas and the rich scattering environment that is often present between the transmitter and the receiver both ensure that the signal reaches the receiver via a large number of independent paths. On the other hand, FSO links are much more directive and, for example, the presence of a small cloud might induce large fades on all source-detector sub-channels simultaneously [3]. Consequently, the high performance gains promised by MIMO-FSO systems might not be achieved in practice and "*alternative means of operation in such environments must be considered*" [3].

On the other hand, cooperative communication is emerging as a new communication paradigm where multiple nodes in a wireless network can cooperate with each other to form a virtual antenna array and profit from the underlying spatial diversity in a distributed manner [6]. Cooperative diversity is based on the broadcast nature of RF transmissions where a message transmitted from a source node can be overheard by neighboring nodes and then can be processed and relayed to the destination node. Consequently, questions arise on the utility of cooperation for the directive LOS FSO networks.

While the literature on cooperation in RF networks is huge and dates back to about a decade [6], it was only recently that some contributions considered this transmission strategy in the context of FSO communications [7], [8]. In [7], a cooperation strategy based on the implementation of convolutional codes was proposed and analyzed and in [8] a cooperation strategy that can be implemented independently from the structure of the channel code was considered. Both contributions showed the utility of cooperation for FSO systems despite the non-broadcast nature of FSO transmissions.

While [7] and [8] were limited to the case of one relay, we propose a novel cooperation strategy that can be applied with any number of relays. We further analyze the performance of the proposed scheme in the presence of shot noise under the assumption that background noise is negligible. This assumption is justified by the fact that diversity techniques are designed to combat fading (and not noise) and they result in the highest performance gains at high signal-to-noise ratios (SNR) [7], [8]. Note that, for low SNRs, it is better not to cooperate since the relays will be forwarding noisy replicas of the information they received [8]. Consequently, for cooperative systems that are designed to operate at high SNRs, the received signal strength is sufficiently large so that the signal-dependent shot noise and fading become the main limiting factors [3]. While [7] and [8] are both limited to the case where the Channel-State-Information (CSI) is not available neither at the transmitter nor at the receiver sides, another contribution of this work resides in investigating the impact of CSI on the performance of cooperative FSO networks. In this context, we propose an optimal power allocation strategy that is based on minimizing a tight upper-bound on the error probability.

II. Cooperation Strategy and System Model

A. Cooperation Strategy

Consider the example of a FSO Metropolitan Area Network with two buildings A and C having several FSO units placed on their top. Each unit consists of an optical transmitter and receiver and is deployed to establish a full-duplex FSO link with a neighboring building. Given the high directivity of FSO transmissions, one separate transceiver is entirely dedicated

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for the communication with a certain neighboring building. Consider also a certain number of neighboring buildings B_1, B_2, ... and assume that two separate FSO links are set up between each one of these buildings and buildings A and C.

For the above scenario, a cooperation protocol can be implemented to achieve spatial diversity if the transceivers on buildings B_1, B_2, ... are willing to cooperate in order to enhance the communication reliability between buildings A and C. This cooperation can be realized by temporarily dedicating the links (A-B_1), (A-B_2), ... and (B_1-C), (B_2-C), ... for relaying the information that A has to communicate with C (or vice vera). By abuse of notation, buildings A and C will be denoted by source S and destination D, respectively, while buildings B_1, $B_2, \ldots$ will be denoted as relays R_1, R_2, ... In what follows, we denote by N_r the number of relays cooperating with S and D. It is worth noting that the transceivers at R_1, ..., R_{N_r} are not deployed with the objective of relaying the data of S. In fact, these transceivers are deployed for R_1, ..., R_{N_r} to communicate with S and D. Now, if R_1, ..., R_{N_r} are willing to share their existing resources (and they have no information to transmit), then they can act as relays for assisting S in its communication with D. Note that in a different communication session, S and D can act as relays for the communication between R_i and R_j.

The cooperation strategy that we propose applies to systems that suffer from shot noise in the absence of background radiation. The transmitted symbols are assumed to be carved from a Q-ary pulse position modulated are implemented at the destination and the relays. In the absence of background radiation, the only source of photons is the information-carrying light signal itself. Consequently, only two scenarios are possible at each receiver: either (i) exactly one slot contains a nonzero count implying that a correct decision can be made or (ii) all slots have a zero count; in this case, deciding randomly in favor of one of the slots will result in a correct decision with probability $1/Q$.

The cooperation strategy that we propose is as follows: at a first time, a sequence of symbols is transmitted from S to D and to the N_r relays. At a second time, each relay decodes its received symbols. If at a certain relay, a nonzero photon count was observed in one slot, then this relay has detected the information symbol correctly and it participates in the cooperation effort by retransmitting this symbol to D. On the other hand, if all counts are equal to zero, then most probably the corresponding relay will make an erroneous decision (with probability $\frac{Q-1}{Q}$). In order to avoid confusing D by forwarding a wrong estimate of the symbol, the relay backs off and stops its retransmission during the corresponding symbol duration. Note that the retransmissions from all cooperating relays occur simultaneously. Given the non-broadcast nature of FSO transmissions, there is no interference between the different FSO units involved in each cooperation cycle. Consequently, no particular coding is required for separating the data streams that are transmitted simultaneously from the relays to D. This justifies the adaptability of the above simple strategy that is based on spatial repetitions for FSO networks. Finally, note that the proposed strategy does not require any kind of CSI and it can be implemented without feedback.

B. System Model

Denote by a_0, $a_{\mathrm{s},1}, \ldots, a_{\mathrm{s},N_r}$ and $a_{1,\mathrm{d}}, \ldots, a_{N_r,\mathrm{d}}$ the path gains of the links S-D, S-$R_1, \ldots,$ S-R_{N_r} and R_1-D, ..., R_{N_r}-D, respectively. In this work, we adopt the lognormal and Rayleigh turbulence-induced fading channel models [3]. In the lognormal model, the probability density function (pdf) of the path gain ($a > 0$) is given by: $f_A(a) = \frac{1}{\sqrt{2\pi}\sigma a} \exp\left(-\frac{(\ln a - \mu)^2}{2\sigma^2}\right)$ where the parameters μ and σ satisfy the relation $\mu = -\sigma^2$ so that the mean path intensity is unity: $\mathrm{E}[I] = \mathrm{E}[A^2] = 1$. The degree of fading is measured by the scintillation index defined by: S.I. $= e^{4\sigma^2} - 1$. Typical values of S.I. range between 0.4 and 1. In the Rayleigh model, the pdf of the path gain ($a > 0$) is: $f_A(a) = 2ae^{-a^2}$.

Denote by P_0 the fraction of the total power that is dedicated to the direct link S-D. In the same way, denote by $P_1^{(n)}$ and $P_2^{(n)}$ the fractions of the total power dedicated to links S-R_n and R_n-D, respectively. In order to ensure the same transmission level as in non-cooperative systems, the following equality must be satisfied: $P_0 + \sum_{n=1}^{N_r}[P_1^{(n)} + P_2^{(n)}] = 1$.

We consider Q-ary PPM with intensity modulation and direct detection (IM/DD) where each receiver corresponds to a simple photoelectrons counter. Denote by λ_s the average number of photoelectrons per slot resulting from the incident light signal. λ_s is given by [3]:

$$\lambda_s = \eta \frac{P_r T_s / Q}{hf} = \eta \frac{E_s}{hf} \tag{1}$$

where η is the detector's quantum efficiency assumed to be equal to 1 in what follows, $h = 6.6 \times 10^{-34}$ is Planck's constant and f is the optical center frequency taken to be 1.94×10^{14} Hz (corresponding to a wavelength of 1550 nm). T_s stands for the symbol duration while P_r stands for the optical power that is incident on the receiver. Finally, $E_s = P_r T_s / Q$ corresponds to the received optical energy per symbol corresponding to the direct link S-D.

Consider first the link S-D and denote by $\mathbf{Z_0} = [Z_{0,1}, \ldots, Z_{0,Q}]$ the Q-dimensional vector whose q-th component corresponds to the number of photoelectrons in the q-th slot. In the absence of background radiation, if the transmitted symbol is $s \in \{1, \ldots, Q\}$, then the decision variable $Z_{0,s}$ can be modeled as a Poisson random variable (r.v.) with parameter $P_0 a_0^2 \lambda_s$ while the remaining $Q-1$ slots will be empty: $Z_{0,q} = 0$ for $q \neq s$.

In the same way, we denote the decision vector observed at the n-th relay by $\mathbf{Z_1^{(n)}} = [Z_{1,1}^{(n)}, \ldots, Z_{1,Q}^{(n)}]$. Given that the symbol s was transmitted simultaneously to the destination and to the relays, then: $Z_{1,q}^{(n)} = 0$ for $q \neq s$ while $Z_{1,s}^{(n)}$ is a Poisson r.v. whose parameter is given by:

$$\mathrm{E}\left[Z_{1,s}^{(n)}\right] = \beta_1^{(n)} P_1^{(n)} a_{\mathrm{s},n}^2 \lambda_s \quad ; \quad n = 1, \ldots, N_r \tag{2}$$

where $\beta_1^{(n)}$ is a gain factor associated with the n-th relay and resulting from the fact that S might be closer to R_n than it is to D. Performing a typical link budget analysis [3] shows that $\beta_1^{(n)} = \left(\frac{d_{SD}}{d_{SR_n}}\right)^2$ where d_{SD} and d_{SR_n} stand for the distances from S to D and from S to R_n, respectively, for $n = 1, \ldots, N_r$.

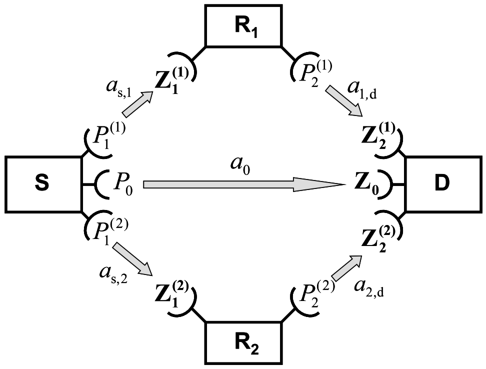

Fig. 1. The cooperation scheme with two relays.

By inspecting the decision vector $\mathbf{Z_1^{(n)}}$, the n-th relay decides in favor of symbol $\hat{s}^{(n)}$ where (for $n = 1, \dots, N_r$):

$$\hat{s}^{(n)} = \arg \max_{q=1,\dots,Q} Z_{1,q}^{(n)} \equiv \arg_{q=1,\dots,Q} \left[Z_{1,q}^{(n)} \neq 0 \right] \tag{3}$$

where the above decision rules are equivalent since, in the absence of background radiation, at least $Q-1$ slots of $\mathbf{Z_1^{(n)}}$ have a zero photon count.

Denote the decision vector at the destination corresponding to the link R$_n$-D by $\mathbf{Z_2^{(n)}} = [Z_{2,1}^{(n)}, \dots, Z_{2,Q}^{(n)}]$. Based on the proposed cooperation strategy, the statistics of the components of $\mathbf{Z_2^{(n)}}$ depend on the decision taken at the n-th relay. If at least one component of $\mathbf{Z_1^{(n)}}$ is different from zero, then a correct decision was made at the n-th relay since in the absence of background radiation the only source of this nonzero count is the presence of a light signal in the corresponding slot. In this case, the n-th relay retransmits the symbol $\hat{s}^{(n)} = s$ along the link R$_n$-D. Consequently, $Z_{2,q}^{(n)} = 0$ for $q \neq s$ while $Z_{2,s}^{(n)}$ is a Poisson r.v. with parameter:

$$\mathrm{E}\left[Z_{2,s}^{(n)} \right] = \beta_2^{(n)} P_2^{(n)} a_{n,\mathrm{d}}^2 \lambda_s \quad ; \quad n = 1, \dots, N_r \tag{4}$$

where $\beta_2^{(n)} = \left(\frac{d_{SD}}{d_{R_n D}} \right)^2$ with $d_{R_n D}$ corresponding to the distance between R$_n$ and D.

On the other hand, if all components of $\mathbf{Z_1^{(n)}}$ are equal to zero, then a correct decision can not be guaranteed at the n-th relay. In this case, the n-th relay stops its transmission (for one symbol duration corresponding to s) implying that $\mathbf{Z_2^{(n)}}$ will be equal to the all-zero vector. The cooperation strategy and the different parameters are depicted in Fig. 1 for $N_r = 2$.

III. Performance Analysis

A. Optical Detection

The decision taken at D will be based on the vectors $\mathbf{Z_0}$, $\mathbf{Z_2^{(1)}}, \dots, \mathbf{Z_2^{(N_r)}}$. The proposed strategy ensures that the nonzero counts in the above vectors (if present) will be all in the same PPM slot. Note that all-zero counts in $\mathbf{Z_2^{(n)}}$ follow from either (i) all-zero counts in $\mathbf{Z_1^{(n)}}$ (implying that the n-th relay will not cooperate with S) or (ii) the n-th relay retransmitted the correct symbol but because of fading and shot noise along the link R$_n$-D, zero photons were observed in the corresponding slot.

Defining the vector $\mathbf{Z}$ as $\mathbf{Z} \triangleq \mathbf{Z_0} + \sum_{n=1}^{N_r} \mathbf{Z_2^{(n)}}$, the decision rule at D is given by:

$$\tilde{s} = \begin{cases} \arg_q [Z_q \neq 0], & \mathbf{Z} \neq \mathbf{0}_Q; \\ \mathrm{rand}(1, \dots, Q), & \mathbf{Z} = \mathbf{0}_Q. \end{cases} \tag{5}$$

where $\mathbf{0}_Q$ corresponds to the Q-dimensional all-zero vector while the function $\mathrm{rand}(1, \dots, Q)$ corresponds to choosing randomly one integer in the set $\{1, \dots, Q\}$.

B. Conditional error probability with one relay

The channel state is defined by the vector $A \triangleq [a_0, a_{\mathrm{s},1}, \dots, a_{\mathrm{s},N_r}, a_{1,\mathrm{d}}, \dots, a_{N_r,\mathrm{d}}]$. For $N_r = 1$ relay, the conditional symbol-error probability (SEP) assuming that the symbol s was transmitted can be written as:

$$P_{e|A} = \Pr(Z_{0,s} > 0) p_1 + \Pr(Z_{0,s} = 0) \Pr(Z_{1,s}^{(1)} = 0) p_2 + \Pr(Z_{0,s} = 0) \Pr(Z_{1,s}^{(1)} > 0) \left[\Pr(Z_{2,s}^{(1)} > 0) p_3 + \Pr(Z_{2,s}^{(1)} = 0) p_4 \right] \tag{6}$$

where $p_1 = 0$ since a nonzero count in slot s along the link S-D implies certainly that the symbol was transmitted in this slot. On the other hand, $p_2 = \frac{Q-1}{Q}$ since when all-zero counts are observed along the link S-R$_1$, the relay does not participate in the retransmission; moreover, when all-zero counts are also observed along the link S-D, then $\mathbf{Z} = \mathbf{0}_Q$ and a random decision is made at D. Now $p_3 = 0$ since $Z_{2,s}^{(1)} > 0$ will imply that $Z_s > 0$ resulting in no error. Finally, $p_4 = \frac{Q-1}{Q}$ since $(Z_{0,s}, Z_{2,s}^{(1)}) = (0, 0)$ will imply that $Z_s = 0$ resulting in a random decision at D. Therefore, eq. (6) can be written as:

$$P_{e|A} = \frac{Q-1}{Q} \Pr(Z_{0,s} = 0) \left[\Pr(Z_{1,s}^{(1)} = 0) + \Pr(Z_{1,s}^{(1)} > 0) \Pr(Z_{2,s}^{(1)} = 0) \right] \tag{7}$$

Note that because of the symmetry of the PPM constellation, $P_{e|A}$ does not depend on the value taken by the symbol s. On the other hand, $\Pr(Z_{0,s} = 0) = e^{-P_0 a_0^2 \lambda_s}$. From eq. (2), $\Pr(Z_{1,s}^{(1)} = 0) = 1 - \Pr(Z_{1,s}^{(1)} > 0) = e^{-\beta_1^{(1)} P_1^{(1)} a_{\mathrm{s},1}^2 \lambda_s}$ and from eq. (4): $\Pr(Z_{2,s}^{(1)} = 0) = e^{-\beta_2^{(1)} P_2^{(1)} a_{1,\mathrm{d}}^2 \lambda_s}$. Replacing these terms in eq. (7) results in:

$$P_{e|A} = \frac{Q-1}{Q} e^{-k_0 P_0} \left[e^{-k_1^{(1)} P_1^{(1)}} + e^{-k_2^{(1)} P_2^{(1)}} - e^{-k_1^{(1)} P_1^{(1)}} e^{-k_2^{(1)} P_2^{(1)}} \right] \tag{8}$$

where the constants k_0 and $\{k_1^{(n)}, k_2^{(n)}\}_{n=1}^{N_r}$ are positive real numbers defined as:

$$k_0 \triangleq a_0^2 \lambda_s \quad ; \quad k_1^{(n)} \triangleq \beta_1^{(n)} a_{\mathrm{s},n}^2 \lambda_s \quad ; \quad k_2^{(n)} \triangleq \beta_2^{(n)} a_{n,\mathrm{d}}^2 \lambda_s \tag{9}$$

Equation (8) shows that there is a two fold increase in the diversity. In fact, $P_{e|A}$ is large when either a_0 and $a_{\mathrm{s},1}$ are both small (the links S-D and S-R$_1$ are both in deep fades) or when a_0 and $a_{1,\mathrm{d}}$ are both small (the links S-D and R$_1$-D are both in deep fades).

C. Conditional error probability with more than one relay

Proposition: In the presence of N_r relays, the conditional SEP can be expressed as the product of $N_r + 1$ terms corresponding to the links S-D, S-R$_1$-D, ..., S-R$_{N_r}$-D as follows:

$$P_{e|A}(N_r) = \frac{Q-1}{Q}e^{-k_0 P_0}.$$
$$\prod_{n=1}^{N_r}\left[e^{-k_1^{(n)}P_1^{(n)}} + e^{-k_2^{(n)}P_2^{(n)}} - e^{-k_1^{(n)}P_1^{(n)}}e^{-k_2^{(n)}P_2^{(n)}}\right] \quad (10)$$

Proof: We will prove the above relation by induction. Eq. (10) reduces to eq. (8) for $N_r = 1$. Assume that the above relation holds for $N_r - 1$ and prove that it holds for N_r.

We define the probability $P'_{e|A}(N_r)$ as: $P'_{e|A}(N_r) = P_{e|A}(N_r)\frac{Q}{Q-1}$. For N_r relays, $P_{e|A}(N_r)$ can be written as:

$$P_{e|A}(N_r) = \left[1 - P'_{e|A}(N_r-1)\right]p_1 + P'_{e|A}(N_r-1)p_2 \quad (11)$$

where $p_1 = 0$ since with probability $1 - P'_{e|A}(N_r-1)$ the system formed from the first $N_r - 1$ relays and corresponding to the set of N_r links S-D, S-R$_1$-D, ..., S-R$_{N_r-1}$-D is providing the destination with at least one decision vector containing a non-zero count. Since the proposed cooperation strategy ensures retransmissions only in the correct slot, then no error is made in this case. On the other hand, with probability $P'_{e|A}(N_r-1)$ the above system of $N_r - 1$ relays is providing the destination with N_r all-zero decision vectors. In this case, the reliability of the transmission will be determined by the link S-R$_{N_r}$-D provided by the N_r-th relay. Consequently, p_2 can be written as:

$$\begin{aligned} p_2 &= \frac{Q-1}{Q}\left[\Pr(Z_{1,s}^{(N_r)} = 0) + \Pr(Z_{1,s}^{(N_r)} > 0)\Pr(Z_{2,s}^{(N_r)} = 0)\right] \\ &= \frac{Q-1}{Q}\left[e^{-k_1^{(N_r)}P_1^{(N_r)}} + \left(1 - e^{-k_1^{(N_r)}P_1^{(N_r)}}\right)e^{-k_2^{(N_r)}P_2^{(N_r)}}\right] \end{aligned} \quad (12)$$

Now substituting p_1 and p_2 by their values in eq. (11) results in eq. (10).

The conditional SEP given in eq. (10) can be bounded by:

$$P_{e|A}(N_r) \leq \frac{Q-1}{Q}e^{-k_0 P_0}\prod_{n=1}^{N_r}\left[e^{-k_1^{(n)}P_1^{(n)}} + e^{-k_2^{(n)}P_2^{(n)}}\right] \quad (13)$$

where this upper-bound becomes tighter for large values of E_s. In fact, asymptotically, the term $e^{-k_1^{(n)}P_1^{(n)}}e^{-k_2^{(n)}P_2^{(n)}}$ is two orders of magnitude smaller than the terms $e^{-k_1^{(n)}P_1^{(n)}}$ and $e^{-k_2^{(n)}P_2^{(n)}}$.

D. Error probability and diversity order

Averaging the conditional SEP given in eq. (10) over the distributions of $a_0, a_{\text{s},1}, \ldots, a_{\text{s},N_r}, a_{1,\text{d}}, \ldots, a_{N_r,\text{d}}$ shows that the SEP can be written under the following form:

$$P_e = \frac{Q-1}{Q}P_{e,0}\prod_{n=1}^{N_r}\left[P_{e,1}^{(n)} + P_{e,2}^{(n)} - P_{e,1}^{(n)}P_{e,2}^{(n)}\right] \quad (14)$$

In the case of Rayleigh fading, $P_{e,0} = (1 + P_0\lambda_s)^{-1}$ and $P_{e,i}^{(n)} = (1 + \beta_i^{(n)}P_i^{(n)}\lambda_s)^{-1}$ for $i = 1, 2$ and $n = 1, \ldots, N_r$. This shows that P_e scales asymptotically as $\lambda_s^{-(N_r+1)}$ (rather than λ_s^{-1} as in 1×1 non-cooperative FSO links). This implies that the proposed cooperation strategy permits to achieve a diversity order of $N_r + 1$ in the presence of N_r relays.

For lognormal fading, the integrals involved in the calculation of the SEP do not admit a closed-form solution. In this case, the different terms in eq. (14) can be written as: $P_{e,0} = \text{Fr}(P_0\lambda_s, 0, \sigma)$ and $P_{e,i}^{(n)} = \text{Fr}(\beta_i^{(n)}P_i^{(n)}\lambda_s, 0, \sigma)$ for $i = 1, 2$ and $n = 1, \ldots, N_r$ where $\text{Fr}(a, 0, b)$ is the lognormal density frustration function defined in [9] as:

$$\text{Fr}(a, 0, b) = \int_0^{+\infty}\frac{1}{\sqrt{2\pi b^2}x}e^{-ax^2}\exp\left[-\frac{(\ln(x) + b^2)^2}{2b^2}\right]\mathrm{d}x \quad (15)$$

IV. Power Allocation in the Presence of CSI

In the absence of CSI, no preference can be made among the available links. In this case, the transmit power must be equally distributed among the $2N_r + 1$ links S-D, S-R$_1$, ..., S-R$_{N_r}$ and R$_1$-D, ..., R$_{N_r}$-D by setting:

$$P_0 = P_1^{(1)} = \cdots = P_1^{(N_r)} = P_2^{(1)} = \cdots = P_2^{(N_r)} = \frac{1}{2N_r + 1} \quad (16)$$

On the other hand, when the path gains are known for a given channel realization, the values of P_0 and $\{P_1^{(n)}, P_2^{(n)}\}_{n=1}^{N_r}$ can be optimized in order to minimize the conditional error probability.

The power allocation strategy that we propose is based on minimizing the upper-bound given in eq. (13) rather than the exact expression of $P_{e|A}$ given in eq. (10) for the following reasons: (i) The minimization of $P_{e|A}$ given in eq. (10) turns out to be tedious and does not result in simple closed-form solutions that lend themselves to feasible implementation in realistic systems. (ii) Diversity techniques achieve their highest performance gains in the high SNR regime and it is in this region that the bound given in eq. (13) becomes extremely close to the exact expression of $P_{e|A}$.

A. Power allocation with one relay

Proposition: The optimal values of $\{P_0, P_1^{(1)}, P_2^{(1)}\}$ that minimize the bound in eq. (13) subject to the constraints $P_0 + P_1^{(1)} + P_2^{(1)} = 1$ and $P_0 \geq 0$, $P_1^{(1)} \geq 0$ and $P_2^{(1)} \geq 0$ are given by:

$$(P_0, P_1^{(1)}, P_2^{(1)}) = \left(0, \frac{k_2^{(1)} + \log\frac{k_1^{(1)}}{k_2^{(1)}}}{k_1^{(1)} + k_2^{(1)}}, \frac{k_1^{(1)} + \log\frac{k_2^{(1)}}{k_1^{(1)}}}{k_1^{(1)} + k_2^{(1)}}\right) \quad (17)$$

if $\frac{1}{k_0} \geq \frac{1}{k_1^{(1)}} + \frac{1}{k_2^{(1)}}$ and $\max(k_1^{(1)}, k_2^{(1)}) \geq \log\frac{\max(k_1^{(1)}, k_2^{(1)})}{\min(k_1^{(1)}, k_2^{(1)})}$ and by:

$$(P_0, P_1^{(1)}, P_2^{(1)}) = (1, 0, 0) \quad (18)$$

otherwise.

Proof: The constrained minimization is based on the method of Lagrange multipliers with the solution satisfying the KKT conditions in order to ensure non-negative powers. A detailed proof is provided in Appendix I.

The general solution given in eq. (17)-(18) shows that the optimal power allocation strategy corresponds to transmitting the entire optical power either along the direct link S-D or along the indirect link S-R$_1$-D depending on which link is "stronger". The condition $\frac{1}{k_0} \geq \frac{1}{k_1^{(1)}} + \frac{1}{k_2^{(1)}}$ shows that the strength of the direct link can be measured by k_0 while the strength of the indirect link can be measured by $(\frac{1}{k_1^{(1)}}+\frac{1}{k_2^{(1)}})^{-1}$. Note that this result is consistent with the findings related to non-cooperative MIMO-FSO systems where the optimal strategy corresponds to transmitting the entire optical power along the strongest path [4]. Note also that the condition $\max(k_1^{(1)}, k_2^{(1)}) \geq \log \frac{\max(k_1^{(1)}, k_2^{(1)})}{\min(k_1^{(1)}, k_2^{(1)})}$ is more easily satisfied for large values of λ_s implying that the indirect link S-R$_1$-D becomes more preferable over the direct link S-D at higher SNRs. At low signal levels, instead of dedicating a certain amount of power to communicate with the relay that will most probably observe all-zero counts and hence will not cooperate, it is better to transmit the entire power along the direct link.

B. Power allocation with more than one relay

In this section, we determine the solution $\underline{P} = (P_0, P_1^{(1)}, P_2^{(1)}, \cdots, P_1^{(N_r)}, P_2^{(N_r)})$ that minimizes eq. (13) for any number of relays.

Define the powers $\{\tilde{P}_1^{(n)}, \tilde{P}_2^{(n)}\}_{n=1}^{N_r}$ as:

$$\tilde{P}_1^{(n)} = \frac{k_2^{(n)} + \log(k_1^{(n)}/k_2^{(n)})}{k_1^{(n)} + k_2^{(n)}} \; ; \; \tilde{P}_2^{(n)} = \frac{k_1^{(n)} + \log(k_2^{(n)}/k_1^{(n)})}{k_1^{(n)} + k_2^{(n)}} \tag{19}$$

where $\tilde{P}_1^{(n)}$ and $\tilde{P}_2^{(n)}$ fall in the interval $[0 \; 1]$ if:

$$\max(k_1^{(n)}, k_2^{(n)}) \geq \log \frac{\max(k_1^{(n)}, k_2^{(n)})}{\min(k_1^{(n)}, k_2^{(n)})} \tag{20}$$

Denote by $\mathcal{N}_f \subset \{1, \dots, N_r\}$ the set of values of n for which eq. (20) holds and denote its cardinality by N_f. In this case, N_f possible candidate solutions to the power allocation problem are given by:

$$\underline{P}_n = \Big(0, (0,0), \cdots, (\tilde{P}_1^{(n)}, \tilde{P}_2^{(n)}), \cdots, (0,0)\Big) \; ; \; n \in \mathcal{N}_f \tag{21}$$

and they correspond to transmitting the total power along one of the paths S-R$_n$-D for $n \in \mathcal{N}_f$. From eq. (13), the error probability corresponding to $\underline{P}_n$ is given by:

$$f_n = \frac{Q-1}{Q}\left[e^{-k_1^{(n)}\tilde{P}_1^{(n)}} + e^{-k_2^{(n)}\tilde{P}_2^{(n)}}\right] \; ; \; n \in \mathcal{N}_f \tag{22}$$

Another candidate solution corresponding to the direct link S-D and its corresponding error probability are given by:

$$\underline{P}_0 = (1, (0,0), \cdots, (0,0)) \quad ; \quad f_0 = \frac{Q-1}{Q} e^{-k_0} \tag{23}$$

Proposition: Among the set of all feasible candidate solutions, the solution that minimizes eq. (13) is given by $\underline{P} = \underline{P}_{\tilde{n}}$ where the integer $\tilde{n}$ is chosen as follows:

$$\tilde{n} = \arg\min \left\{ \{f_0\} \cup \left\{ f_n \mid \frac{1}{k_0} \geq \frac{1}{k_1^{(n)}} + \frac{1}{k_2^{(n)}} \; ; \; n \in \mathcal{N}_f \right\} \right\} \tag{24}$$

Once again, a path selection algorithm must be implemented according to eq. (24) in order to transmit the total optical power either along the direct path S-D or along one of the indirect paths S-R$_n$-D for $n \in \mathcal{N}_f$. For FSO cooperative systems, this path selection approach turns out to be optimal.

Proof: We will prove the above proposition by induction. The above strategy reduces to that given in section IV-A for $N_r = 1$. Assume that it holds for a network with less than N_r relays and prove its optimality for a network with N_r relays.

Assume that in the optimal solution there is at least one value of n for which $P_1^{(n)} = P_2^{(n)} = 0$. In this case, at least one relay is turned off (not cooperating) and the system reduces to a system having less than N_r relays. In this case, the optimal solution is as given in eq. (24) following from the assumption made on the optimality with less than N_r relays. The remaining possibilities are either (i) all components of $\underline{P}$ are different from zero (the power is distributed among the links S-D, S-R$_1$-D, ..., S-R$_{N_r}$-D) or (ii) the first component of $\underline{P}$ is equal to zero while the remaining components are different from zero (the power is distributed among the links S-R$_1$-D, ..., S-R$_{N_r}$-D). In Appendix II we prove that such solutions are not optimal implying that the optimal power allocation strategy is as given in eq. (24).

V. Numerical Results

We next present some numerical results that support the theoretical claims made in the previous sections. For simulation purposes, we assume that $\beta_1^{(n)}$ (resp. $\beta_2^{(n)}$) is the same for all values of n implying that all relays are at the same distance from the source (resp. destination). These values will be denoted by β_1 and β_2, respectively. For small number of relays where numerical optimization is possible, results showed that the proposed power allocation strategy is extremely close to the optimal strategy where the power ratios are determined numerically from minimizing the exact value of the conditional error probability (rather than minimizing the upper-bound).

Fig. 2 shows the performance of 4-PPM over Rayleigh fading channels in the absence of CSI. Results show the high performance levels and the enhanced diversity orders achieved by the proposed scheme. Even in the worst case of $\beta_1 = \beta_2 = 1$ ($d_{SR_n} = d_{R_nD} = d_{SD}$ for all values of n), cooperation with one relay results in a performance gain of about 8 dB at a SEP of 10^{-3}. In this case, cooperation is useful for values of E_s exceeding -175 dBJ. As (β_1, β_2) increases from $(1, 1)$ to $(4, 4)$, the value of E_s above which cooperation is useful drops to about -185 dBJ. This figure also shows the excellent match between simulations and the exact SEP expression in eq. (14). Similar results are obtained in Fig. 3 in the presence of CSI. In this case, performance gains are achieved over the entire range of E_s. For one relay at a SEP of 10^{-3}, the availability of CSI results in additional gain of about 3 dB compared to the no-CSI case.

Fig. 4 shows the variation of the SEP as a function of the number of relays for $E_s = -170$ dBJ in the case of lognormal fading with S.I. $= 0.6$. The presence of only one relay that is relatively close to S and D (in particular, $\beta_1 = \beta_2 = 4$) and the selection of the best link among S-D and S-R-D can ensure an extremely small error probability in the order of 10^{-8}.

VI. Conclusion

We investigated the utility of user cooperation as a fading-mitigation technique for FSO networks. In the absence of CSI, the optical power can be evenly distributed among the different links and high performance gains can be achieved at

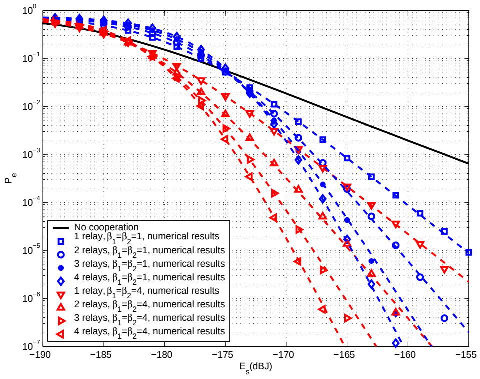

Fig. 2. Performance of 4-PPM in the absence of CSI over Rayleigh fading channels. The dashed lines correspond to the exact SEP given in eq. (14).

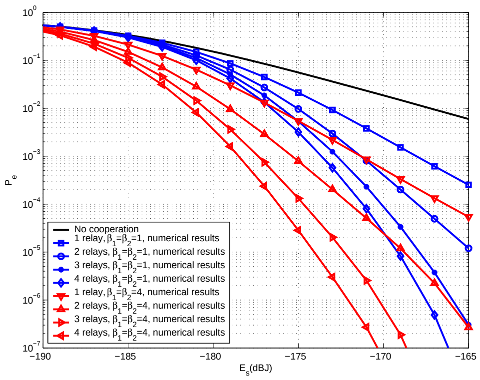

Fig. 3. Performance of 4-PPM in the presence of CSI over Rayleigh fading channels.

signal energies that are not very large. In the presence of CSI, the analytical minimization of an upper-bound on the error probability showed that the best performance can be achieved by transmitting the entire power along the strongest link.

APPENDIX I

In this appendix, we drop the superscripts of $k_1^{(1)}$, $k_2^{(1)}$, $P_1^{(1)}$ and $P_2^{(1)}$ for notational simplicity. Minimizing eq. (13) is equivalent to minimizing the function $f(P_0, P_1, P_2) = e^{-k_0 P_0}\left[e^{-k_1 P_1} + e^{-k_2 P_2}\right]$. In order to take the equality constraint $\sum_{m=0}^{2} P_m = 1$ and the inequality constraints $-P_m \leq 0$ (for $m = 0, 1, 2$) into consideration, we construct the Lagrangian function:

$$\mathcal{L}(\underline{P}, \lambda, \underline{\mu}) = e^{-k_0 P_0}\left[e^{-k_1 P_1} + e^{-k_2 P_2}\right] + \lambda\left(\sum_{m=0}^{2} P_m - 1\right) - \sum_{m=0}^{2} \mu_m P_m \quad (25)$$

At the point (P_0, P_1, P_2) minimizing $f(P_0, P_1, P_2)$, $\mathcal{L}$ must satisfy the following equations:

$$\frac{\partial \mathcal{L}}{\partial P_0} = -k_0 e^{-k_0 P_0}\left[e^{-k_1 P_1} + e^{-k_2 P_2}\right] + \lambda - \mu_0 = 0 \quad (26)$$

$$\frac{\partial \mathcal{L}}{\partial P_1} = -k_1 e^{-k_0 P_0} e^{-k_1 P_1} + \lambda - \mu_1 = 0 \quad (27)$$

$$\frac{\partial \mathcal{L}}{\partial P_2} = -k_2 e^{-k_0 P_0} e^{-k_2 P_2} + \lambda - \mu_2 = 0 \quad (28)$$

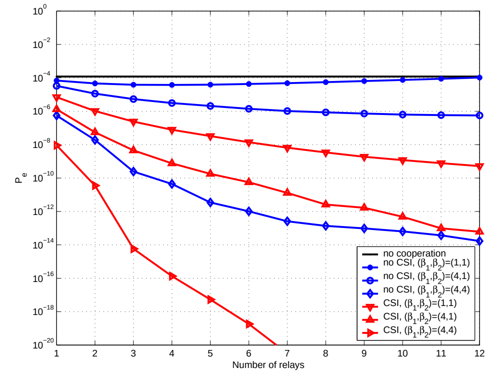

Fig. 4. Performance of 4-PPM for $E_s = -170$ dBJ over lognormal fading channels with S.I. $= 0.6$.

Moreover, the differentiation of $\mathcal{L}$ with respect to the Lagrange multiplier λ implies that:

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \sum_{m=0}^{2} P_m - 1 = 0 \quad (29)$$

The final solution must satisfy the KKT conditions that can be summarized as a set of 3 equalities and 3 inequalities as follows:

$$\mu_m P_m = 0 \quad ; \quad m = 0, \dots, 2 \quad (30)$$

$$\mu_m \geq 0 \quad ; \quad m = 0, \dots, 2 \quad (31)$$

Note that when $P_1 = 0$ then P_2 must be equal to zero and vice versa. Consequently, the general solution can take one of the three following forms.

Case 1: Assume that $P_0 \neq 0$ and $P_1 = P_2 = 0$. In this case, eq. (29) implies that $P_0 = 1$ resulting in $\mu_0 = 0$ following from eq. (30). Replacing P_0, P_1, P_2 and μ_0 by their values in eq. (26) results in $\lambda = 2k_0 e^{-k_0}$. Substituting this value of λ in eq. (27) and eq. (28) results in $\mu_1 = (2k_0 - k_1)e^{-k_0}$ and $\mu_2 = (2k_0 - k_2)e^{-k_0}$. Consequently, the inequalities $\mu_1 \geq 0$ and $\mu_2 \geq 0$ following from eq. (31) will hold if and only if $2k_0 \geq k_1$ and $2k_0 \geq k_2$. These two inequalities can be combined into the following inequality: $\frac{1}{k_0} \leq \frac{1}{k_1} + \frac{1}{k_2}$.

Consequently, the optimal solution takes the form $(P_0, P_1, P_2) = (1, 0, 0)$ when $\frac{1}{k_0} \leq \frac{1}{k_1} + \frac{1}{k_2}$.

Case 2: Assume that $P_0 = 0$ while $P_1 \neq 0$ and $P_2 \neq 0$. In this case, eq. (30) implies that $\mu_1 = \mu_2 = 0$ resulting in $\lambda = k_1 e^{-k_1 P_1} = k_2 e^{-k_2 P_2}$ following from eq. (27) and eq. (28). Combining this equation with the equality $P_1 + P_2 = 1$ that follows from eq. (29) and solving for P_1 and P_2 results in:

$$P_1 = \frac{k_2 + \log(k_1/k_2)}{k_1 + k_2} \quad ; \quad P_2 = \frac{k_1 + \log(k_2/k_1)}{k_1 + k_2} \quad (32)$$

we observe that the solution given in the previous equation is feasible when $k_1 \geq \log \frac{k_1}{k_2}$ and $k_2 \geq \log \frac{k_2}{k_1}$. It is then straight forward to prove that these inequalities are equivalent to the inequality $\max(k_1, k_2) \geq \log \frac{\max(k_1,k_2)}{\min(k_1,k_2)}$. Now solving eq. (26) for μ_0 results in:

$$\mu_0 = \lambda - k_0 e^{-k_1 P_1} - k_0 e^{-k_2 P_2} \quad (33)$$

$$= \lambda - k_0 \frac{\lambda}{k_1} - k_0 \frac{\lambda}{k_2} = k_0 \lambda \left(\frac{1}{k_0} - \frac{1}{k_1} - \frac{1}{k_2}\right) \quad (34)$$

given that $k_0 \geq 0$ and $\lambda \geq 0$ (since $\lambda = k_1 e^{-k_1 P_1}$ with $k_1 \geq 0$), then the inequality $\mu_0 \geq 0$ that follows from eq. (31) can be satisfied if and only if $\frac{1}{k_0} \geq \frac{1}{k_1} + \frac{1}{k_2}$.

Consequently, the optimal solution takes the form given in eq. (32) along with $P_0 = 0$ when $\frac{1}{k_0} \geq \frac{1}{k_1} + \frac{1}{k_2}$ and $\max(k_1, k_2) \geq \log \frac{\max(k_1,k_2)}{\min(k_1,k_2)}$.

Case 3: Assume that $P_0 \neq 0$, $P_1 \neq 0$ and $P_2 \neq 0$. In this case, eq. (30) implies that $\mu_0 = \mu_1 = \mu_2 = 0$. Replacing these values in eq. (26)-(28) results in $\frac{\lambda}{k_0} = e^{-k_0 P_0} e^{-k_1 P_1} + e^{-k_0 P_0} e^{-k_2 P_2}$, $\frac{\lambda}{k_1} = e^{-k_0 P_0} e^{-k_1 P_1}$ and $\frac{\lambda}{k_2} = e^{-k_0 P_0} e^{-k_2 P_2}$, respectively. These three equalities imply that $\frac{1}{k_0} = \frac{1}{k_1} + \frac{1}{k_2}$ (note that $\lambda \neq 0$ since $k_0, \ldots, k_2$ and $P_0, \ldots, P_2$ are all finite).

Since the equality $\frac{1}{k_0} = \frac{1}{k_1} + \frac{1}{k_2}$ among the parameters $k_0, \ldots, k_2$ that depend on the random path gains does not hold in general, then the optimal solution can not take the form considered under case 3. As a conclusion, only case 1 and case 2 are feasible. Note that even when $\frac{1}{k_0} \geq \frac{1}{k_1} + \frac{1}{k_2}$, the inequality $\max(k_1, k_2) \geq \log \frac{\max(k_1,k_2)}{\min(k_1,k_2)}$ might not be satisfied implying that one term among P_1 and P_2 given in eq. (32) will be negative while the second term will be greater than one. If this is the case, then the solution taking the form considered under case 2 can not be optimal and the only remaining feasible solution will be that considered under case 1.

APPENDIX II

For notational simplicity define the parameters $P_1, \ldots, P_{2N_r}$ by: $P_i = P_1^{(\lceil i/2 \rceil)}$ if i is odd and $P_i = P_2^{(\lceil i/2 \rceil)}$ if i is even so that vector $\underline{P}$ can be written as: $\underline{P} = (P_0, P_1, \ldots, P_{2N_r})$. Define the scalars $k_1, \ldots, k_{2Nr}$ in the same way so that the bound in eq. (13) is proportional to the function $F \triangleq e^{-k_0 P_0} \prod_{i=1 \ ; \ i \text{ odd}}^{2N_r - 1} [e^{-k_i P_i} + e^{-k_{i+1} P_{i+1}}]$. Construct the Lagrangian:

$$\mathcal{L}(\underline{P}, \lambda, \underline{\mu}) = e^{-k_0 P_0} \prod_{i=1 \ ; \ i \text{ odd}}^{2N_r - 1} \left[e^{-k_i P_i} + e^{-k_{i+1} P_{i+1}} \right] + \lambda (\sum_{m=0}^{2N_r} P_m - 1) - \sum_{m=0}^{2N_r} \mu_m P_m \quad (35)$$

$\mathcal{L}$ must satisfy the following equation:

$$\frac{\partial \mathcal{L}}{\partial P_0} = -k_0 e^{-k_0 P_0} \prod_{i=1 \ ; \ i \text{ odd}}^{2N_r - 1} \left[e^{-k_i P_i} + e^{-k_{i+1} P_{i+1}} \right] + \lambda - \mu_0 = 0 \quad (36)$$

as well as the following $2N_r$ equations (for $j = 1, \ldots, 2N_r$):

$$\frac{\partial \mathcal{L}}{\partial P_j} = -k_j e^{-k_0 P_0} e^{-k_j P_j} \prod_{j \neq i = 1 ; i \text{ odd}}^{2N_r - 1} \left[e^{-k_i P_i} + e^{-k_{i+1} P_{i+1}} \right] + \lambda - \mu_j = 0 \quad (37)$$

The KKT conditions can be summarized as a set of $2N_r + 1$ equalities and inequalities as follows:

$$\mu_m P_m = 0 \ ; \quad m = 0, \ldots, 2N_r \quad (38)$$

$$\mu_m \geq 0 \ ; \quad m = 0, \ldots, 2N_r \quad (39)$$

We need to consider the following two cases:

Case 1: Assume that $P_0 = 0$ and $P_m \neq 0$ for $m = 1, \ldots, 2N_r$. In this case, eq. (38) implies that $\mu_m = 0$ for $m = 1, \ldots, 2N_r$. Substituting P_0 and μ_1 by their values in eq. (37) for $j = 1$ results in $\lambda = k_1 e^{-k_1 P_1} \prod_{i=3}^{2N_r - 1} \left[e^{-k_i P_i} + e^{-k_{i+1} P_{i+1}} \right]$. This implies that λ is positive and different from zero since $k_1, \ldots, k_{2N_r}$ and $P_1, \ldots, P_{2N_r}$ are all finite (note that, being a continuous random variable, k_1 is different from 0).

Let m be an odd integer in the set $\{1, \ldots, 2N_r - 1\}$. Replacing $\mu_m = 0$ in eq. (37) for $j = m$ and $\mu_{m+1} = 0$ in eq. (37) for $j = m + 1$ and adding up the obtained equations results in:

$$\lambda \left(\frac{1}{k_m} + \frac{1}{k_{m+1}} \right) = F \ ; \ m \in \{1, \ldots, 2N_r - 1\} \ ; \ m \text{ odd} \quad (40)$$

Since this equality holds for any value of m and since $\lambda \neq 0$, then eq. (40) implies that $\frac{1}{k_1} + \frac{1}{k_2} = \cdots = \frac{1}{k_{2N_r - 1}} + \frac{1}{k_{2N_r}}$. Since there is no guarantee that these N_r equalities will hold (since $k_1, \ldots, k_{2N_r}$ depend on the random path gains), then the solution can not take the form considered under this case.

Case 2: Assume that $\underline{P} \neq (0, \ldots, 0)$. In this case, eq. (38) implies that $\mu_m = 0$ for $m = 0, \ldots, 2N_r$. Since eq. (40) follows from $\mu_1, \ldots, \mu_{2Nr}$ being all equal to zero, then this equation will hold in this second case as well. On the other hand, replacing $\mu_0 = 0$ in eq. (36) and solving for λ results in:

$$\frac{\lambda}{k_0} = F \quad (41)$$

Combining eq. (40) and eq. (41) results in the following $N_r + 1$ equalities: $\frac{1}{k_0} = \frac{1}{k_1} + \frac{1}{k_2} = \cdots = \frac{1}{k_{2N_r - 1}} + \frac{1}{k_{2N_r}}$. Since the above equalities among the random variables $k_0, \ldots, k_{2N_r}$ do not hold in general, then the optimal solution can not take the form $\underline{P} \neq (0, \ldots, 0)$.

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