On the Achievable Diversity Orders over Non-Severely Faded Lognormal Channels

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***Abstract*— We propose to approximate the distribution of the sum of identically-distributed lognormal random variables by an Erlang distribution. The advantage of the proposed approximation over the lognormal and non-lognormal approximations proposed in the literature resides mainly in the fact that it results in simple closed-form expressions of the average bit-error-rate (BER) thus offering clear insights on the performance of diversity combining techniques over correlated and uncorrelated lognormal fading channels. For instance, the proposed approach shows that, for typical values of the BER, the diversity order in the case of non-severe lognormal fading can be accurately approximated by the order of the approximating Erlang distribution.**

***Index Terms*— Lognormal-sum distribution, Erlang distribution, lognormal fading, diversity, performance analysis.**

I. Introduction

Recent research focused on evaluating the performance of diversity combining techniques over lognormal fading channels. The interest in the lognormal distribution arises from the fact that it accurately models the propagation over indoor channels [1] and over turbulent atmospheric Free-Space Optical (FSO) channels [2]. In the same way, the IEEE 802.15.3a channel model recommendation adopted lognormal cluster and multi-path shadowing for characterizing the Ultra-Wideband (UWB) channels [3]. In this context, the outage probability of SIMO-RF systems employing Selective-Combining (SC) was studied in [4]. [5] provided expressions for the outage probability of MRC, EGC and SC. [6] investigated the average Bit-Error-Rate (BER) performance over MIMO-FSO links. Finally, the average BER and the achievable diversity gain over UWB channels were analyzed in [7] and [8] respectively.

Despite this extensive effort in analyzing the lognormal fading channels, an intuitive and simple closed-form expression that characterizes the performance of diversity combining techniques over such channels is unavailable. For example, over Rayleigh fading channels, the average BER scales asymptotically as SNR^{-L} where SNR stands for the signal-to-noise ratio and L corresponds to the number of parallel channels between the transmitter and the receiver; however, a similar expression for lognormal fading channels is still missing. For example, [4]- [5] necessitate the numerical evaluation of L and $L-1$ nested integrals, respectively. In the same way, in [6]- [7], the average BER was expressed in integral forms that do not admit closed-form solutions since they correspond to variants of the lognormal density frustration function defined in [9] as: $\mathrm{Fr}(a,0,b) = \int_0^{+\infty} \frac{1}{\sqrt{2\pi}bx} \exp\left(-ax^2\right) \exp\left[-\frac{(\ln(x)+b^2)^2}{2b^2}\right] \mathrm{d}x$. Finally, despite the fact that the integrals in [5]- [6] were evaluated numerically by applying the Gauss-Legendre quadrature formula, the obtained results do not give a clear idea about how do the outage probability and BER scale with the SNR.

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Evaluating the BER over lognormal fading channels requires a knowledge of the Probability Density Function (PDF) of the sum of lognormal random variables. The literature of approximating the lognormal-sum PDF that does not have an exact expression is huge [10]–[13]. The lognormal-sum distribution is often approximated by another lognormal distribution by applying different methods and optimization techniques [10]. Recently non-lognormal approximations were considered where a two-component mixture of lognormals [11], a power lognormal distribution [12] and a type IV Pearson distribution [13] were proposed. However, the disadvantage of approximating the lognormal-sum PDF by a lognormal PDF resides in the fact that the average BER will have an integral form that does not admit a closed-form solution [7]. On the other hand, the complexity of the approaches adopted in [11]–[13] results in complicated expressions of the PDFs and in intractable results in terms of the BER.

In this paper, we propose the Erlang distribution as an alternative approximation for the lognormal-sum distribution in the case of "non-severe" fading; that is, in the case where the variance of the summands is relatively small. Despite the fact that the proposed Erlang approximation is not as accurate as the other existing and widely approved approximations [10]–[13], the motivations behind adopting the Erlang distribution can be summarized as follows (1): The Erlang distribution results in closed-form PDFs with acceptable accuracy for a wide range of lognormal summands and their corresponding parameters, (2): the Erlang distribution holds in both cases of the sum of correlated or uncorrelated lognormal random variables, (3): the Erlang approximation results in simple closed-form asymptotic expressions of the average BER with the lognormal and lognormal-sum distributions.

II. Lognormal-sum approximation

A. Preliminaries

Consider the case of L parallel lognormal fading channels that are described by the channel coefficients $\{h_l\}_{l=1}^{L}$. Assuming that these channels are identically distributed, the PDF of the lognormal random variable (r.v.) h_l can be written as:

$$f(h_l) = \frac{\xi}{\sqrt{2\pi}\sigma h_l} \exp\left[-\frac{\left(20\log_{10}(h_l)-\mu\right)^2}{2\sigma^2}\right] \qquad (1)$$

where μ and σ^2 correspond to the mean and variance of the Gaussian r.v. $20\log_{10}(h_l)$ for all values of l and ξ is a constant that is given by: $\xi = 20/\ln(10)$. The average power of h_l is normalized to unity by fixing $\mu = -\sigma^2/\xi$.

For a given channel realization, the instantaneous signal to noise ratio is proportional to the r.v. h defined as: $h = \sum_{l=1}^{L} h_l^2$. Since $h_1, \ldots, h_L$ are lognormal r.v.s, then $h_1^2, \ldots, h_L^2$ are also lognormal r.v.s implying that h corresponds to the summation of L lognormal random variables.

The average BER using BPSK is given by:

$$P_e = \int_0^{\infty} Q\left(\sqrt{SNR.x}\right) f_h(x)\mathrm{d}x \tag{2}$$

where $f_h(x)$ corresponds to the PDF of the r.v. h and the function $Q(x)$ is defined by: $Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^{\infty} e^{-t^2/2}\mathrm{d}t$.

In this paper, we are interested in the asymptotic behavior of P_e for $SNR \gg 1$. Since $Q(x) \leq \frac{1}{\sqrt{2\pi}}\frac{e^{-x^2/2}}{x}$:

$$P_e \leq \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\frac{1}{\sqrt{SNR.x}}\exp\left[-\frac{SNR.x}{2}\right] f_h(x)\mathrm{d}x \tag{3}$$

with the above upper-bound becoming tighter for larger values of the SNR. Note that the above integral can not be evaluated analytically when the lognormal-sum PDF $f_h(.)$ is approximated by any one of the distributions proposed in [10]–[13].

B. Erlang approximation of the lognormal-sum PDF

In this work, we propose to approximate the exact lognormal-sum PDF $f_h(x)$ by an Erlang PDF $g_h(x)$:

$$g_h(x) = \frac{e^{-\lambda x}x^{n-1}\lambda^n}{(n-1)!} \quad ; \quad x \geq 0,\ n \in \mathbb{N},\ \lambda > 0 \tag{4}$$

The main motivation behind adopting the Erlang distribution as our desired generic approximation resides first in its simplicity and in the resemblance between its PDF and that of the lognormal-sum distribution. On the other hand, in their analysis of queues, the authors of [14] suggested an Erlang distribution in the case where the squared coefficient of variation $c_v^2 = \frac{\mathrm{var}[h]}{E^2[h]}$ of the service distribution does not exceed 0.5. Since we observed that c_v^2 is always less than 0.5 for the case of non-severe fading, this constituted the second motivation in considering the Erlang approximation.

In this work, the choice of n and λ is based on the criterion of minimizing the mean-square-error $MSE = \int_0^{+\infty}[f_h(x) - g_h(x)]^2\mathrm{d}x$ between the exact and approximate distributions. This solution can be obtained numerically by applying the least-squares nonlinear curve fitting techniques. The convergence and accuracy of such algorithms can be enhanced by an appropriate choice of the starting point. Matching the means and variances of $f_h(.)$ and $g_h(.)$ results in:

$$E_f[h] = E_g[h] = \frac{n}{\lambda} \quad ; \quad \mathrm{var}_f[h] = \mathrm{var}_g[h] = \frac{n}{\lambda^2} \tag{5}$$

where the functions $E_f[.]$ and $E_g[.]$ (resp. $\mathrm{var}_f[.]$ and $\mathrm{var}_g[.]$) correspond to evaluating the means (resp. variances) with respect to the exact and approximate distributions $f_h(.)$ and $g_h(.)$ respectively. Solving the above equations results in: $(n_0, \lambda_0) \triangleq (n, \lambda) = \left(\frac{E_f^2[h]}{\mathrm{var}_f[h]}, \frac{E_f[h]}{\mathrm{var}_f[h]}\right)$ where $E_f[h]$ and $\mathrm{var}_f[h]$ can be evaluated numerically from the exact distribution. The point (n_0, λ_0) is chosen as the starting point of the algorithm.

Results show that the Erlang approximation is accurate for values of σ not exceeding 3 dB. In other words, the proposed approach is useful only for evaluating the performance over channels that do not suffer from severe fading. Practically, this corresponds to the case of FSO links with non-severe turbulence [2] or to indoor UWB channels [3].

C. BER calculation

After finding the optimal Erlang PDF $g_h(.)$ that best fits $f_h(.)$, we replace $f_h(x)$ by its approximation in eq. (3). The integral in this equation can be readily solved resulting in:

$$P_e \leq \frac{1}{\sqrt{2\pi}}\frac{\Gamma(n-\frac{1}{2})}{(n-1)!}\frac{1}{\left(\frac{SNR}{\lambda}\right)^{\frac{1}{2}}}\frac{1}{\left(1+\frac{\mathrm{SNR}}{2\lambda}\right)^{n-\frac{1}{2}}} \tag{6}$$

where $\Gamma(x) = \int_0^{+\infty} e^{-t}t^{x-1}\mathrm{d}t$ corresponds to the Gamma function ($x \geq 0$).

For sufficiently large values of the SNR, the upper-bound in eq. (6) becomes tight implying that P_e can be approximated in the high SNR regime by:

$$P_e \approx \frac{2^{n-1}}{\sqrt{\pi}}\frac{\Gamma(n-\frac{1}{2})}{(n-1)!}\frac{1}{\left(\frac{SNR}{\lambda}\right)^{n}} \tag{7}$$

Equation (7) shows that P_e scales asymptotically as $\left(\frac{SNR}{\lambda}\right)^{-n}$ implying that the overall diversity order is equal to n which is nothing but the order of the Erlang approximation. Inspecting eq. (7) also shows that the dominant term that has the major influence on the BER performance is the parameter n. While the slope of the BER curve changes with n, the term $1/\lambda$ only results in an additional shift of the BER curve. Consequently, the parameter n alone can constitute a simple quantitative indicator on the achievable asymptotic diversity orders over lognormal fading channels. Interestingly, the numerical results show that n is an increasing function of L, a decreasing function of σ and a decreasing function of ρ (the correlation coefficient between the different co-channels).

On the other hand, for finite SNRs, the diversity order of lognormal fading channels can be quantified in a more meaningful way by the so-called *relative diversity order* defined in [15] as:

$$RDO(SNR) = \left(\frac{\partial \log P_e}{\partial \log SNR}\right)\left(\frac{\partial \log P_{e,BM}}{\partial \log SNR}\right)^{-1} \tag{8}$$

where P_e is given in eq. (6) and $P_{e,BM}$ is the average BER of a benchmark scheme corresponding to $L = 1$ [15]. Note that, from eq. (7), $RDO(SNR)$ is proportional to n as $SNR \gg 1$.

III. Numerical Results

We performed an extensive numerical analysis that showed that the proposed approximation is accurate for all values of L, σ and ρ in the intervals $\{1, \ldots, 16\}$, $[0\ 3]$ dB and $[0\ 1]$ respectively. For example, Fig. 1 shows that the exact and approximate PDFs are very close for $\sigma = 2$ dB and $\rho = 0$.

Fig. 2 shows the close match between the exact and approximate BERs that are evaluated numerically from eq. (2) using the exact and approximate PDFs, respectively, for $\sigma = 2$ dB and $\rho = 0$. This figure also shows the upper-bound given

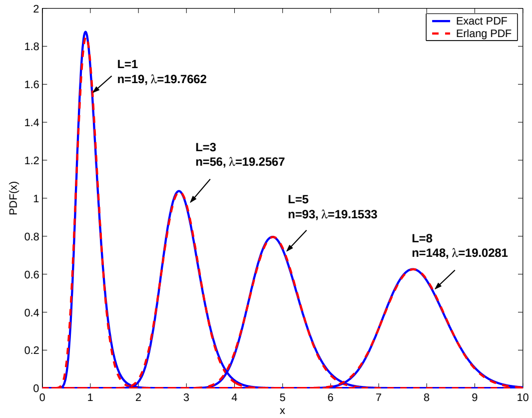

Fig. 1. Exact PDFs versus the approximate PDFs for $\sigma = 2$ dB and $\rho = 0$.

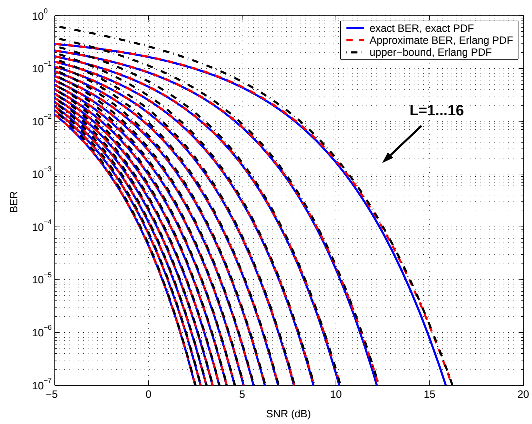

Fig. 2. BER performance for $\sigma = 2$ dB and $\rho = 0$.

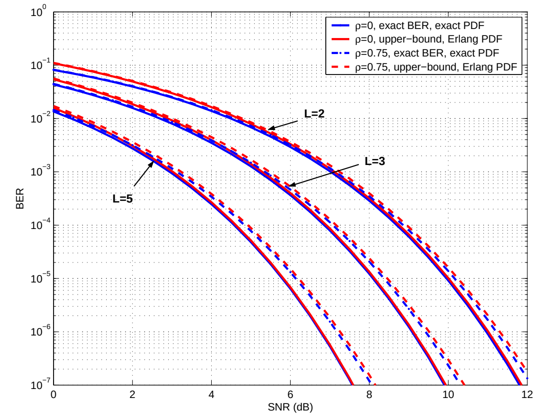

Fig. 3. Impact of correlation on the BER performance for $\sigma = 1.5$ dB.

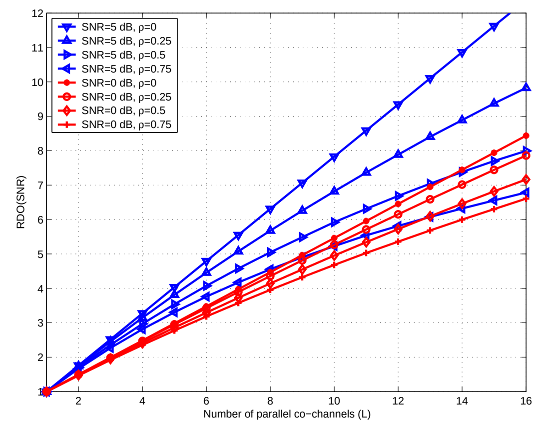

Fig. 4. The relative diversity order $RDO(SNR)$ for $\sigma = 2$ dB.

in eq. (7). The proposed upper-bound turns out to be very close to the exact BER for SNRs that are not excessively very large. Similar results are obtained in Fig. 3 in the presence of channel correlation ($\rho = 0.75$). This figure also shows the performance loss that results from channel correlation.

Fig. 4 shows the variation of $RDO(SNR)$ as a function of L for different values of SNR. Results clearly show the increase of the diversity order with L and its decrease with ρ.

IV. Conclusion

We investigated the performance of diversity combining techniques over identically-distributed non-severely faded log-normal channels. We have shown that the average BER scales asymptotically as SNR^{-n} where n stands for the order of the Erlang distribution used to approximate the exact lognormal or lognormal-sum distribution.

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