Space-Time Coding For Multiuser Ultra-Wideband Communications

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***Abstract*— In this paper, we present the construction of full rate, fully diverse and totally real space-time (ST) codes for ultra-wideband (UWB) transmissions. In particular, we construct two families of codes adapted to real carrier-less UWB communications that employ Pulse Position Modulation (PPM), Pulse Amplitude Modulation (PAM) or a combination of the two. The first family encodes adjacent symbols and is constructed from totally real cyclic division algebras. The second family encodes the pulses used to convey one information symbol and permits to achieve high performance levels with reduced complexity. The first family of codes achieves only a fraction of the coding gain of the second one. Moreover, these coding gains are independent from the size of the transmitted constellation. For Time-Hopping (TH) multiple access channels, the amplitude spreading code associated with the second family of codes is taken to be user-specific. In this case, a simple design criterion is proposed and spreading matrices constructed according to this criterion permit to reduce the level of multiple access interference (MAI). Simulations performed over realistic indoor UWB channels verify the theoretical claims and show high performance levels and better immunity against MAI.**

***Index Terms*— Space-Time coding, ultra-wideband (UWB), multiple input multiple output (MIMO), pulse amplitude modulation (PAM), Rake, multiple access interference.**

I. INTRODUCTION

Recently Ultra-wideband (UWB) emerged as a strong candidate for a wide variety of indoor wireless applications. One solution to the growing demand of such applications for high-rate reliable communications is to combine UWB with Multiple-Input-Multiple-Output (MIMO) techniques [1]–[4]. For a system equipped with P transmit antennas, [1], [2] achieve a transmit diversity order of P while transmitting at a rate of one symbol per channel use (PCU) while [3], [4] achieve a rate of P symbols PCU without any transmit diversity gain. In the literature many full rate (P symbols PCU) and fully diverse codes were proposed [5]–[8]. However, unlike conventional communications, UWB operates at the baseband level rendering the application of these complex-valued coding techniques impossible. Apart from this constraint, the time-hopping (TH) UWB has an appealing feature that resides in the transmission of a train of pulses for conveying one information symbol. We will show later that taking advantage of this property results in the design of totally real UWB-specific space-time (ST) codes.

In this paper, we propose the construction of two families of UWB-specific ST codes. The first family is based on the adaptation of the construction techniques presented in [7], [8] to real valued situations. The second family encodes the pulses used to convey one symbol and is specific to TH systems. We then extend this scheme to multiple access scenarios. It was noticed in [9] that using randomly polarized pulses can ameliorate multiple access in TH-UWB even in asynchronous environments. We further investigate this point and propose the construction of well designed unitary matrices that result in a better interference rejection at the receiver.

The rest of the paper is organized as follows. Section II describes the system model of the MIMO TH-UWB systems. The constructions of the different coding schemes are presented in section III. In section IV, we consider the problem of multiple access interference and we propose a simple and efficient method for the construction of the amplitude spreading matrices. An analysis of the performance of multiuser MIMO-UWB systems is also presented in this section. Numerical results are presented in section V while section VI concludes.

Notations: $I_{m\times n}$ denotes the first m columns of the $n\times n$ identity matrix I_n. $\mathbb{1}_{m\times n}$ and $\mathbb{0}_{m\times n}$ correspond to the $m\times n$ matrices whose elements are all equal to 1 and 0 respectively. $*$ stands for convolution, $\otimes$ for the Kronecker product and $\mathbb{Q}$ denotes the field of rational numbers. The functions $\mathrm{N}_{\mathbb{K}/\mathbb{Q}}$ and $\mathrm{Tr}_{\mathbb{K}/\mathbb{Q}}$ denote the norm and trace of an element in the field extension $\mathbb{K}/\mathbb{Q}$. $\text{diag}(X_1,\dots,X_n)$ corresponds to stacking the corresponding matrices on the principal diagonal. $\text{vec}(X)$ stands for stacking the columns of the matrix X vertically.

II. SYSTEM MODEL

Constructed from M-ary pulse position modulated (PPM) signals, a M-PPM-M'-PAM signal set is obtained by including M' pulse amplitude modulated (PAM) signals at each position. Each symbol in the constructed set is therefore defined by its amplitude and position coordinates $(a,d)\in\{(2m'-1-M')\ ;\ m'=1,\dots,M'\}\times\{0,\dots,M-1\}$ and can be represented by the M-dimensional vector $[a_0,\dots,a_{M-1}]^T$ where $a_m=a\delta(d-m)$ corresponds to the amplitude of the signal transmitted at the m-th position. The corresponding transmitted pulse is given by $aw(t-d\delta)=\sum_{m=0}^{M-1}a_m w(t-m\delta)$ where $w(t)$ is the UWB transmitted pulse waveform of duration T_w and normalized in order to have unit energy. δ is the modulation delay with $\delta\geq T_w$ in order to avoid the overlapping between consecutive positions at the transmitter

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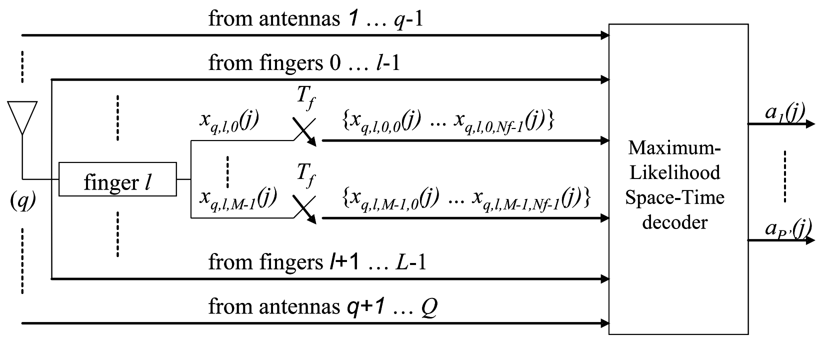

Fig. 1. Schematic representation of the receiver structure during the j-th symbol duration. The output of the l-th Rake finger consists of M-decision variables corresponding to the M modulation positions for $l = 0, \ldots, L-1$. The space-time decoder consists of despreading the received frames and performing Maximum Likelihood detection. $P' = P$ for uncoded systems and IPC (Section III-B) while $P' = P^2$ for ISC (Section III-A).

side. However, these positions interfere with each other at the receiver side because of the multi-path propagation.

In TH-UWB systems, each information symbol is conveyed by N_f pulses and the signal transmitted from the p-th antenna of the k-th user can be expressed as:

$$s_p^{(k)}(t) = \sqrt{\frac{E_k}{PN_f}} \sum_{n=0}^{N_f-1} \sum_{m=0}^{M-1} a_{p,m}^{(k)} b_{p,n}^{(k)} w(t - nT_f - c_k(n)T_c - m\delta) \tag{1}$$

where E_k is the relative transmit energy of the k-th user with respect to the first user ($E_0 = 1$) and it is normalized by P to insure the same total transmitted energy as in single-antenna systems. The frame duration T_f includes N_c slots of duration T_c with $T_c \geq T_w$ and $N_c T_c \leq T_f$. All the transmit antennas of the k-th user will share the same pseudo-random time hopping code $c_k(n) \in \{0, \ldots, N_c - 1\}$. The $P \times N_f$ matrix $B^{(k)}$, whose $(p, n+1)$-th element is equal to $b_{p,n}^{(k)}$ for $p = 1, \ldots, P$ and $n = 0, \ldots, N_f - 1$, stands for the amplitude spreading code associated with the k-th user. These matrices introduce an additional coding between the different users and between the data streams transmitted by different antennas of the same user.

The received signal at the q-th antenna of a receiver equipped with Q antennas is given by:

$$r_q(t) = \sum_{k=0}^{K} \sum_{p=1}^{P} s_p^{(k)}(t) * g_{q,p}^{(k)}(t - \tau^{(k)} - \varepsilon_{q,p}^{(k)}) + n_q(t) \tag{2}$$

$$= \frac{1}{\sqrt{PN_f}} \sum_{p,n,m} a_{p,m} b_{p,n} h_{q,p}(t - nT_f - c(n)T_c - m\delta) + \underbrace{\frac{1}{\sqrt{PN_f}} \sum_{k=1}^{K} \sqrt{E_k} I_{MAI}^{(k)}}_{MAI} + n_q(t) \tag{3}$$

where the superscript was skipped for the first user and MAI stands for the multiple access interference. The interference from the k-th user is given by:

$$I_{MAI}^{(k)} = \sum_{p,n,m} a_{p,m}^{(k)} b_{p,n}^{(k)} h_{q,p}^{(k)}(t - nT_f - c_k(n)T_c - m\delta - \tau^{(k)}) \tag{4}$$

In eq. (2), $n_q(t)$ is the noise at the q-th antenna assumed to be real AWGN with variance $N_0/2$. $\tau^{(k)}$ corresponds to the propagation delay of the k-th user relative to the first user. $g_{q,p}^{(k)}(t)$ stands for the impulse response of the frequency selective channel between the p-th transmit antenna of the k-th user and the q-th receive antenna. $\varepsilon_{q,p}^{(k)}$ corresponds to the time delay of the first arriving multi-path component of $g_{q,p}^{(k)}(t)$. In fact, given the very short duration of the transmitted pulses, the spacing between the different antennas introduces propagation delays that can be comparable with T_w. These additional delays are calculated with respect to the first arriving ray of the first user. In other words, $\min_{q,p}(\varepsilon_{q,p}^{(0)}) = 0$. In what follows, we fix $h_{q,p}^{(k)}(t) = w(t) * g_{q,p}^{(k)}(t - \varepsilon_{q,p}^{(k)})$ implying that the receiver is synchronized to the first multi-path component between the transmit and the receive arrays. Inter Symbol Interference (ISI) can be eliminated by choosing $T_f \geq \max_{q,p,k}(T_{q,p}^{(k)}) + (M-1)\delta + N_c T_c + T_w$, where $T_{q,p}^{(k)}$ is the maximum delay spread of $g_{q,p}^{(k)}(t)$.

In order to take advantage of the multi-path diversity with moderate complexity, a simplified L-th order Rake receiver [10] is adopted by choosing the finger delays as $\Delta_l = lMT_w$ for $l = 0, \ldots, L-1$. Moreover, at each arm of the Rake receiver, a bank of M correlators is needed to detect the M-dimensional signals resulting in a total of QML decision variables collected during each time frame. Ignoring MAI for the moment, these decision variables are given by:

$$x_{q,l,m,n} = \int_{nT_f}^{(n+1)T_f} r_q(t) \tilde{w}_{l,m,n}(t) \mathrm{d}t \tag{5}$$

where $\tilde{w}_{l,m,n}(t) = w(t - nT_f - c(n)T_c - \Delta_l - m\delta)$. Following from the condition of no ISI, the decision variables in eq. (5) take the following form:

$$x_{q,l,m,n} = \frac{1}{\sqrt{PN_f}} \sum_{p',m'} a_{p',m'} b_{p',n} r_{q,p'}\left((m - m')\delta + \Delta_l\right) + n_{q,l,m,n} \tag{6}$$

where $r_{q,p}(\tau) = \int_0^{T_f} h_{q,p}(t) w(t - \tau) \mathrm{d}t$. A schematic representation of the multi-antenna Rake receiver is given in Fig. 1. The noise term $n_{q,l,m,n} = \int_{nT_f}^{(n+1)T_f} n_q(t) \tilde{w}_{l,m,n}(t) \mathrm{d}t$ is

Gaussian. The correlation between the noise samples verifies:

$$E[n_{q,l,m,n}n_{q',l',m',n'}] = \frac{N_0}{2} \gamma\left((m'-m)\delta + \Delta_{l'} - \Delta_l\right)\delta(q'-q)\delta(n'-n) \quad (7)$$

where $\gamma(t)$ is the autocorrelation function of $w(t)$. Choosing $\delta \geq T_w$ and $\Delta_l = lMT_w$ results in a white Gaussian noise since $\gamma(kT_w) = 0$ for all nonzero integer values of k.

Equation (6) can be expressed in matrix form as:

$$X_{q,l}(j) = \frac{1}{\sqrt{PN_f}} R_{q,l} A(j) B + N_{q,l}(j) \quad (8)$$

where B is the $P \times N_f$ amplitude spreading matrix. The $M \times N_f$ matrices $X_{q,l}(j)$ and $N_{q,l}(j)$ are composed from the decision and noise terms respectively. The decision variables are collected during the j-th symbol duration $[(j-1)\ j]N_f T_f$. Denoting by $a_p(j) = [a_{p,0}(j), \ldots, a_{p,M-1}(j)]^T$ the vector representation of the symbol transmitted by the p-th antenna during the j-th symbol duration, then the $MP \times P$ matrix $A(j)$ can be expressed as $A(j) = \text{diag}\left(a_1(j), \ldots, a_P(j)\right)$. Finally, $R_{q,l} = [R_{q,l,1}, \ldots, R_{q,l,P}]$ is the $M \times PM$ channel matrix. $R_{q,l,p}$ is a $M \times M$ matrix whose (m, m')-th element is equal to $r_{q,p}(\Delta_l + (m-m')\delta)$. In what follows, the factor $1/\sqrt{PN_f}$ is disregarded since it can be included in the noise variance.

From eq. (8), and for space-time codewords that extend over J symbol durations, stacking the decision matrices corresponding to the different receive antennas and Rake fingers vertically and the matrices corresponding to different symbol durations horizontally results in:

$$X_{(QLM \times JN_f)} = R_{(QLM \times PM)} A_{(PM \times JP)} (I_J \otimes B_{(P \times N_f)}) + N_{(QLM \times JN_f)} \quad (9)$$

where $A = [A(1)\ \cdots\ A(J)]$ and the subscripts indicate the corresponding matrices' dimensions. The noise matrix N is constructed in the same way as the decision matrix X.

III. CODES CONSTRUCTIONS

In what follows, we propose the ST codes constructions in the case where no Channel State Information (CSI) is available at the transmitter side. We propose ST codes for mobile terminals having a maximum number of 6 antennas. For the construction of $P \times J$ ST codes based on the above model, we differentiate between two kinds of systems. The first one encodes adjacent symbols without introducing any particular coding between the pulses used to convey a given symbol. The other one introduces inter-pulse coding. Formally speaking, the first case corresponds to fixing $B = \mathbb{1}_{P \times N_f}$ while B is chosen to verify $BB^T = N_f I_P$ in the second case. In what follows, we limit ourselves to PAM. The extension to hybrid PPM-PAM constellations is discussed later.

A. Inter-Symbol Coding (ISC)

From eq. (9), combining the decision variables corresponding to the different pulses of the same symbol is equivalent to calculating:

$$C_1 = A(I_J \otimes B)(I_J \otimes B)^T = N_f A(I_J \otimes \mathbb{1}_{P,P}) = N_f C \otimes \mathbb{1}_{1,P} \quad (10)$$

where C is the $P \times J$ ST codeword. It is obvious that C and C_1 have the same rank and the same coding gain (up to a scaling factor). So the construction of ST-codes in this situation is equivalent to the classical construction of C (rank and determinant criteria from [11]). An additional constraint related to the nature of the carrier-less transmissions of TH-UWB systems is imposed resulting in totally real codes.

In this subsection, we consider $n \times n$ codewords constructed from cyclic division algebras with $n = P$. First, we start by reminding the basic principles of such constructions (interested readers can refer to [6], [8] for more details). Consider the totally real cyclic number field extension $\mathbb{K}/\mathbb{Q}$ with Galois group $Gal(\mathbb{K}/\mathbb{Q}) = \langle\sigma\rangle$ with $\sigma^n = 1$. The cyclic algebra $\mathcal{A} = (\mathbb{K}/\mathbb{Q}, \sigma, \gamma)$ can be decomposed as $\mathcal{A} = \mathbb{K} \oplus z\mathbb{K} \oplus \cdots \oplus z^{n-1}\mathbb{K}$ where $z \in \mathcal{A}$ verifies $kz = z\sigma(k)$ for all $k \in \mathbb{K}$ and $z^n = \gamma \in \mathbb{Q}^*$.

Theorem 1 [6] : If γ is chosen such that there are no elements in $\mathbb{K}^*$ whose norms are equal to γ^t for $t = 1, \ldots, n-1$, then $\mathcal{A}$ is a division algebra.

In other words, the ST code constructed from the matrix representation of the elements of $\mathcal{A}$ will be fully diverse since all the non-zero elements of $\mathcal{A}$ are invertible. Limiting the construction in the ring of integers $\mathcal{O}_\mathbb{K}$ of $\mathbb{K}$, the $n \times n$ codewords can be expressed as:

$$C_0 = \begin{bmatrix} k_0 & k_1 & k_2 & \cdots & k_{n-1} \\ \gamma\sigma(k_{n-1}) & \sigma(k_0) & \sigma(k_1) & \cdots & \sigma(k_{n-2}) \\ \gamma\sigma^2(k_{n-2}) & \gamma\sigma^2(k_{n-1}) & \sigma^2(k_0) & \cdots & \sigma^2(k_{n-3}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma\sigma^{n-1}(k_1) & \gamma\sigma^{n-1}(k_2) & \gamma\sigma^{n-1}(k_3) & \cdots & \sigma^{n-1}(k_0) \end{bmatrix} \quad (11)$$

where $k_i \in \mathcal{O}_\mathbb{K}$ for $i = 0, \ldots, n-1$. As $\mathbb{K}$ can be viewed as an n-dimensional vector space over $\mathbb{Q}$, then n information symbols are contained in each value of k_i resulting in a full rate code. In other words, $k_i = \sum_{j=0}^{n-1} a_{ni+j+1}\theta^j$ for $i = 0, \ldots, n-1$ where $\{1, \theta, \ldots, \theta^{n-1}\}$ is an integral basis of $\mathcal{O}_\mathbb{K}$ and $a_1, \ldots, a_{n^2}$ are the information symbols that belong to a PAM constellation.

Proposition 1 [8] : If γ is chosen to be an integer, then $\det(C_0)$ is also an integer implying that the minimum of the absolute value of the determinant of all nonzero codewords C_0 is equal to 1 independently from the size of the transmitted constellation.

Consider the n information symbols $a_{in+1}, \ldots, a_{(i+1)n}$ for a given value of $i \in \{0, \ldots, n-1\}$. From eq. (11), these symbols are contained uniquely in the conjugates of k_i according to:

$$\begin{bmatrix} k_i & \cdots & \sigma^{n-1}(k_i) \end{bmatrix}^T = \Gamma(\theta) \begin{bmatrix} a_{in+1} & \cdots & a_{(i+1)n} \end{bmatrix}^T \quad (12)$$

where $\Gamma(\theta)$ is the $n \times n$ matrix whose (i, j)-th element is equal to $\sigma^{i-1}(\theta^{j-1})$ for $i, j \in \{1, \ldots, n\}$.

Therefore, a modified version of the initial n-dimensional extension of the PAM constellation is transmitted. In order not to introduce any distortions on the PAM constellation, it is interesting that the transmitted and the original signal sets keep the same shape. This can be done by multiplying $\Gamma(\theta)$ by a diagonal matrix D such that the matrix $D' = D\Gamma(\theta)$ is unitary.

In this case, the transmitted constellation is a rotated version of the initial signal set and no shaping losses are introduced. If D' is unitary, then $\det(D) = \frac{1}{\sqrt{d_{\mathbb{K}}}}$ since $\det(\Gamma(\theta)\Gamma(\theta)^T) \triangleq d_{\mathbb{K}}$ (the absolute discriminant of $\mathbb{K}$).

When $d_{\mathbb{K}}$ is a perfect square, we take $D = \mathrm{diag}\left(\alpha, \ldots, \sigma^{n-1}(\alpha)\right)$ where $\alpha \in \mathbb{K}$ verifying $\det(D) = \mathrm{N}_{\mathbb{K}/\mathbb{Q}}(\alpha) = \frac{1}{\sqrt{d_{\mathbb{K}}}}$. In this case, the codewords are given by:

$$C = \mathrm{diag}(\alpha, \sigma(\alpha), \ldots, \sigma^{n-1}(\alpha))C_0 \tag{13}$$

When $d_{\mathbb{K}}$ is not a perfect square, we choose a totally positive element $\alpha \in \mathbb{K}$ such that the diagonal matrix D is given by $D = \mathrm{diag}\left(\sqrt{\alpha}, \ldots, \sqrt{\sigma^{n-1}(\alpha)}\right)$ and $\mathrm{N}_{\mathbb{K}/\mathbb{Q}}(\alpha) = \frac{1}{d_{\mathbb{K}}}$. In this case, the codewords take the form:

$$C = \mathrm{diag}\left(\sqrt{\alpha}, \sqrt{\sigma(\alpha)}, \ldots, \sqrt{\sigma^{n-1}(\alpha)}\right) C_0 \tag{14}$$

The above cases correspond to choosing an element α that verifies the following relation:

$$\mathrm{vol}\left(\alpha\mathcal{O}_{\mathbb{K}}\right) = 1 \tag{15}$$

where $\Lambda(\alpha\mathcal{O}_{\mathbb{K}})$ is the lattice generated by the principal ideal $\alpha\mathcal{O}_{\mathbb{K}}$ and $\mathrm{vol}\left(\alpha\mathcal{O}_{\mathbb{K}}\right)$ stands for the volume of the fundamental parallelotope of $\Lambda(\alpha\mathcal{O}_{\mathbb{K}})$.

If γ is chosen to verify theorem 1 and proposition 1, then:

$$d_{\infty}(C) \triangleq \min_{a \in \mathbb{Z}^{n^2},\ a \neq \mathbf{0}_{1\times n^2}} |\det(C)| = \frac{1}{\sqrt{d_{\mathbb{K}}}} \tag{16}$$

Equations (13) and (14) can be also expressed as:

$$C = \sum_{i=0}^{n-1} \mathrm{diag}\left(\mathcal{M}[a_{in+1}, \ldots, a_{(i+1)n}]^T\right)\Omega^i \tag{17}$$

where $\mathcal{M}$ is a $n \times n$ matrix. Let $\theta_i = \sigma^i(\theta)$ for $i = 0, \ldots, n-1$. When $d_{\mathbb{K}}$ is a perfect square, the (i,j)-th element of $\mathcal{M}$ is given by $\sigma^{i-1}(\alpha\theta^{j-1}) = \sigma^{i-1}(\alpha)\theta_{i-1}^{j-1}$. Otherwise, $\mathcal{M}(i,j) = \sqrt{\sigma^{i-1}(\alpha)}\theta_{i-1}^{j-1}$. The matrix Ω has dimensions $n \times n$ and it is given by:

$$\Omega = \begin{bmatrix} \mathbf{0}_{(n-1)\times 1} & I_{n-1} \\ \gamma & \mathbf{0}_{1\times(n-1)} \end{bmatrix} \tag{18}$$

As a conclusion, ISC is based on eq. (17) and the design problem is reduced to the correct choice of the extension field $\mathbb{K}$, of an integer γ verifying theorem 1 and of $\alpha \in \mathbb{K}$ verifying eq. (15). ST codes constructed from eq. (17) will achieve full rate and full diversity with a non-vanishing minimum determinant and with no shaping losses.

As indicated in [12], [13], the choice $|\gamma| = 1$ results in energy-efficient codes having high performance levels. For totally-real codewords, transmitting a uniform average energy per antenna can be realized uniquely by $\gamma = \pm 1$. This implies that the constraint of having totally-real codewords results in energy non-efficient codes for $n \geq 3$ since $\gamma^2 = 1$ is always a norm in $\mathbb{K}$ ($\mathrm{N}_{\mathbb{K}/\mathbb{Q}}(1) = 1$). Therefore, energy-efficient codes from cyclic division algebras are possible for $n = 2$ transmit antennas uniquely. We will now present the construction of the first family of totally-real ST codes for $n = 2, \ldots, 6$.

1) 2×2 codes: Since there is no 2-dimensional field extension having a perfect square discriminant, the code is constructed from eq. (14). The matrix $\mathcal{M}$ from eq. (17) takes the form:

$$\mathcal{M} = \begin{bmatrix} \sqrt{\alpha} & 0 \\ 0 & \sqrt{\sigma(\alpha)} \end{bmatrix} \begin{bmatrix} 1 & \theta \\ 1 & \sigma(\theta) \end{bmatrix} \tag{19}$$

It is possible to have a rotated version of the initial signal set if $\mathcal{M}$ is unitary resulting in:

$$\begin{cases} \alpha\left(1+\theta^2\right) = \sigma(\alpha)\left(1+\sigma(\theta)^2\right) = 1 \\ \sqrt{\alpha\sigma(\alpha)}\left(1+\theta\sigma(\theta)\right) = 0 \end{cases} \tag{20}$$

The second condition is verified if there exists an element θ such that $\theta\sigma(\theta) = \mathrm{N}_{\mathbb{K}/\mathbb{Q}}(\theta) = -1$ and, consequently, $\mathbb{K}$ must have an element whose norm is equal to -1. Therefore, for 2×2 codes from cyclic division algebras, energy efficiency comes at the expense of shaping losses and vice versa. Therefore, a compromise must be made for two transmit antennas.

a) Choose $\gamma = -1$. In this case $\mathbb{K}$ must be chosen to have no element whose norm is equal to -1. It is shown in the appendix that $\mathbb{K} = \mathbb{Q}(\sqrt{3})$ can be a good choice. The constructed code is balanced since the same average energy is transmitted from each antenna during the two symbol durations. The disadvantage is that the transmitted constellation is not a rotation of the initial signal set. The codewords are given by:

$$C_{2,eq} = \frac{1}{2}\begin{bmatrix} a_1 + \sqrt{3}a_2 & a_3 + \sqrt{3}a_4 \\ -(a_3 - \sqrt{3}a_4) & a_1 - \sqrt{3}a_2 \end{bmatrix} \tag{21}$$

Since this codeword can be written as $C_{2,eq} = \frac{1}{2}C_0$, then $d_{\infty}(C_{2,eq}) = \frac{1}{4}$.

b) Construct a rotated constellation by choosing a field having units whose norms are equal to -1. For example, we can choose $\mathbb{K} = \mathbb{Q}(\sqrt{5})$ whose ring of integers is given by $\mathcal{O}_{\mathbb{K}} = \mathbb{Z}(\theta)$ where $\theta = (1+\sqrt{5})/2$. Using KANT software [14], we find that the ideal $2\mathcal{O}_{\mathbb{K}}$ is prime. This proves that there is no element in $\mathbb{K}$ having a norm equal to 2. Therefore, choosing $\gamma = 2$ verifies theorem 1. In this case, the codewords take the form:

$$C_{2,rot} = \begin{bmatrix} \sqrt{\alpha}(a_1 + a_2\theta) & \sqrt{\alpha}(a_3 + a_4\theta) \\ 2\sqrt{\alpha_1}(a_3 + a_4\theta_1) & \sqrt{\alpha_1}(a_1 + a_2\theta_1) \end{bmatrix} \tag{22}$$

where $\alpha_1 = \sigma(\alpha)$ and $\theta_1 = \sigma(\theta) = \frac{1-\sqrt{5}}{2}$. The choice $\alpha = \frac{1}{1+\theta^2} = \frac{3-\theta}{5}$ results in a unitary matrix $\mathcal{M}$ in eq. (19). Since α verifies eq. (15), then $d_{\infty}(C_{2,rot}) = \frac{1}{\sqrt{d_{\mathbb{K}}}}$ with $d_{\mathbb{K}} = 5$.

2) 3×3 code: The construction is performed in $\mathbb{K} = \mathbb{Q}(\theta)$ with $\theta = 2\cos(\frac{2\pi}{7})$ having a discriminant of $d_{\mathbb{K}} = 7^2$. The ideal $2\mathcal{O}_{\mathbb{K}}$ is prime. This proves that the first power of 2 which is a norm of some elements in $\mathbb{K}$ is 2^n. Therefore, choosing $\gamma = 2$ results in a division algebra. Choosing $\alpha = \beta/7$ with $\beta = 1 - \theta + 2\theta^2$ verifies eq. (15) since $\mathrm{N}_{\mathbb{K}/\mathbb{Q}}(\beta) = 7^2$ and so $\mathrm{N}_{\mathbb{K}/\mathbb{Q}}(\alpha) = \frac{\mathrm{N}_{\mathbb{K}/\mathbb{Q}}(\beta)}{7^3} = \frac{1}{\sqrt{d_{\mathbb{K}}}}$. The basis $\{\alpha, \alpha\theta, \alpha\theta^2\}$ is not unitary. Applying the change of basis:

$$T = \begin{bmatrix} -1 & 1 & 1 \\ 3 & 0 & -1 \\ 2 & -1 & -1 \end{bmatrix} \tag{23}$$

we obtain the new unitary basis $7\{v_1, v_2, v_3\} = \{u, \sigma(u), \sigma^2(u)\}$ with $u = -2 + 2\theta + 3\theta^2$ and

$\text{Tr}_{\mathbb{K}/\mathbb{Q}}(v_i v_j) = \delta_{i,j}$ for $i,j = 1,2,3$. The 3×3 code can be constructed from eq. (17) where the (i,j)-th element of $\mathcal{M}$ is given by $\sigma^{i-1}(v_j) = \sigma^{i-1}(\sigma^{j-1}(u)) = (-2+2\theta_k+3\theta_k^2)/7$ with $k = (i+j-2) \bmod 3$ and $\theta_i = 2\cos(\frac{2\pi(i+1)}{7})$ for $i = 0,1,2$.

3) 4×4 code: Let $\mathbb{K} = \mathbb{Q}(2\cos(\frac{2\pi}{15}))$ which has a discriminant of $d_{\mathbb{K}} = 3^2 5^3$. $\gamma = 2$ verifies theorem 1 since the ideal $2\mathcal{O}_{\mathbb{K}}$ is prime. Since $d_{\mathbb{K}}$ is not a perfect square, the code is constructed from eq. (14). Using KANT software [14], we find that the prime factorizations of the ideals $3O_{\mathbb{K}}$ and $5O_{\mathbb{K}}$ are given by:

$$3O_{\mathbb{K}} = (\beta_3 O_{\mathbb{K}})^2 \quad ; \quad \beta_3 = -2+3\theta+\theta^2-\theta^3$$
$$5O_{\mathbb{K}} = (\beta_5 O_{\mathbb{K}})^4 \quad ; \quad \beta_5 = 1+\theta$$

Choosing $\beta = \beta_3\beta_5/15$ verifies eq. (15) since $\text{N}_{\mathbb{K}/\mathbb{Q}}(\beta) = \frac{1}{d_{\mathbb{K}}}$, but a problem arises since the first and second conjugates of β are negative. Using KANT, we find the unit $e = -1 + 4\theta - \theta^3$ whose conjugates have the same sign as those of β and whose norm is equal to 1. Therefore, choosing $\alpha = \beta e = (5+6\theta-\theta^2-2\theta^3)/15$ results in a totally positive element verifying eq. (15). Once again, the rotation matrix $\mathcal{M}$ whose (i,j)-th element is given by $\sqrt{\sigma^{i-1}(\alpha)}\sigma^{i-1}(\theta^{j-1})$ is not unitary and a change of basis must be performed. Denote by $\mathcal{G} = \mathcal{M}\mathcal{M}^T$ the Gramm matrix of $\mathcal{M}$, it is easy to find that the (i,j)-th element of $\mathcal{G}$ is given by $\text{Tr}_{\mathbb{K}/\mathbb{Q}}(\alpha\theta^{i-1}\theta^{j-1})$. Now, an orthogonal basis can be obtained by applying the Lenstra-Lenstra-Lovász (LLL) reduction algorithm [15] on $\mathcal{G}$. This basis is given by $\sqrt{\alpha}\{v_i\}_{i=1}^4 = \sqrt{\alpha}\{1, -1-3\theta+\theta^2+\theta^3, -1-2\theta+\theta^2+\theta^3, -1+3\theta-\theta^3\}$. It is easy to verify that the (i,j)-th element of the Gramm matrix associated with this new basis is equal to $\text{Tr}_{\mathbb{K}/\mathbb{Q}}(\alpha v_i v_j) = \delta_{i,j}$ for $i,j = 1,\ldots,4$. Finally, the codewords are constructed from eq. (17) where the (i,j)-th element of $\mathcal{M}$ is given by $\sqrt{\sigma^{i-1}(\alpha)}\sigma^{i-1}(v_j)$ with $\sigma^i(\theta) = 2\cos(\frac{2\pi(i+1)}{15})$ for $i = 1,2,3$.

4) 5×5 code: This code is built from $\mathbb{K} = \mathbb{Q}(2\cos(\frac{2\pi}{11}))$ whose integral basis and discriminant are given by $\{\theta_i = 2\cos(\frac{2\pi(i+1)}{11})\}_{i=0}^4$ and $d_{\mathbb{K}} = 11^4$ respectively. Once again we can choose $\gamma = 2$ since the ideal $2\mathcal{O}_{\mathbb{K}}$ is prime. So there are no elements in $\mathbb{K}$ having norms equal to γ^t for $t = 1,\ldots,4$. The code is constructed according to eq. (17) where the (i,j)-th element of the 5×5 matrix $\mathcal{M}$ is given by $\sigma^{i-1}(v_j)$ with $11\{v_i\}_{i=1}^5 = \{u, \sigma(u), \sigma^4(u), -\sigma^2(u), -\sigma^3(u)\}$ where $u = 4+2\theta+2\theta^2-\theta^4$ whose norm is equal to 11^3 which verifies eq. (15) since $\text{vol}(\Lambda(u\mathcal{O}_{\mathbb{K}})) = 11^5$.

5) 6×6 code: Let $\mathbb{K} = \mathbb{Q}(2\cos(\frac{2\pi}{13}))$ which has a discriminant of $d_{\mathbb{K}} = 13^5$. Once again, we can choose $\gamma = 2$ since the ideal $2\mathcal{O}_{\mathbb{K}}$ is prime. Using KANT, we find that: $13O_{\mathbb{K}} = (\beta O_{\mathbb{K}})^6$ where $\beta = 2+\theta-\theta^2$. The conjugates of the unit $e = [0,2,2,-3,-1,1]$ have the same sign as those of β; therefore $\alpha = \beta e/13$ is a totally positive element verifying eq. (15) since $\text{N}_{\mathbb{K}/\mathbb{Q}}(\alpha) = \frac{\text{N}_{\mathbb{K}/\mathbb{Q}}(\beta)\text{N}_{\mathbb{K}/\mathbb{Q}}(e)}{13^6} = \frac{13\times 1}{13^6} = \frac{1}{d_{\mathbb{K}}}$. An orthogonal basis is obtained from $\{\theta^i\}_{i=0}^5$ by using the matrix:

$$T = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 & 0 \\ -1 & 6 & 1 & -5 & 0 & 1 \\ 3 & -5 & -1 & 5 & 0 & -1 \\ 1 & -8 & 0 & 6 & 0 & -1 \\ 0 & 6 & -3 & -5 & 1 & 1 \\ -1 & -3 & 3 & 4 & -1 & -1 \end{bmatrix} \tag{24}$$

The new basis is given by $[v_1, v_2, \ldots, v_6]^T = T[1, \theta, \ldots, \theta^5]^T$. The code is constructed from eq. (17) where the (i,j)-th element of $\mathcal{M}$ is given by $\sqrt{\sigma^{i-1}(\alpha)}\sigma^{i-1}(v_j)$ with $\sigma^i(\theta) = 2\cos(\frac{2\pi(i+1)}{13})$.

6) Balanced codes: The transmitted energy of eq. (11) (and consequently that of eq. (17)) can be distributed in a more balanced way among the different transmit antennas and symbol durations resulting in a balanced version C_0' [16]. C_0' is constructed in the same way as eq. (11) by removing γ from the lower triangular part of C_0 and by replacing the constituent elements k_i by $\gamma^{\frac{i}{n}}k_i$ for $i = 0,\ldots,n-1$. This will be referred to as "energy" balancing and the properties of eq. (11) and eq. (17) are conserved since C_0 and C_0' have equal determinants.

Even when "energy" balancing is performed and since $|\gamma| > 1$, the amplitudes attributed to each value of k_i (and its conjugates) are not the same and, consequently, the n symbols contained in the conjugates of k_i for different values of i are not equally protected against the error events. For example the symbols $a_1,\ldots,a_n$ are the most vulnerable while the biggest portion of the energy is used to transmit symbols $a_{n(n-1)},\ldots,a_{n^2}$. An additional kind of balancing (referred to as "error" balancing hereafter) can be performed when N_f is a multiple of n according to:

$$C^{(j)} = \sum_{i=0}^{n-1} \text{diag}\left(\mathcal{M}[a_{[(i+(j-1))n+1]_{n^2}}, \ldots, a_{[(i+j)n]_{n^2}}]^T\right)\Omega^i \tag{25}$$

$$C_{bal} = \begin{bmatrix} C^{(1)} & \cdots & C^{(n)} \end{bmatrix} \tag{26}$$

where $[x]_n = (x-1) \bmod n + 1$. In what follows, energy balancing will be applied systematically to all the constructed codes while the subscript "bal" will be reserved to "error" balancing. For example, the error-balanced version of eq. (22) is given by:

$$D\begin{bmatrix} a_1+a_2\theta & \sqrt{2}(a_3+a_4\theta) & a_3+a_4\theta & \sqrt{2}(a_1+a_2\theta) \\ \sqrt{2}(a_3+a_4\theta_1) & a_1+a_2\theta_1 & \sqrt{2}(a_1+a_2\theta_1) & a_3+a_4\theta_1 \end{bmatrix} \tag{27}$$

where $D = \text{diag}(\sqrt{\alpha}, \sqrt{\alpha_1})$.

7) Remarks: For $n \neq 4$, the procedure used to choose an element α verifying eq. (15) coincides with that used to construct the fully-diverse rotation matrices in [17]. These matrices were used to construct the Threaded Algebraic Space-Time (TAST) codes in [5]. Taking this fact as well as the structure of C_0' into account, we conclude that the constructed codes have the same structure as the TAST codes but now real Diophantine numbers (in contrast to complex ones in [5]) are used to render the different layers of each codeword transparent to each other. The utility of following the design procedure presented in this section resides in the fact that we showed that $\{1, \gamma^{\frac{1}{n}}, \ldots, \gamma^{\frac{n-1}{n}}; \quad \gamma = 2\}$ is in fact the best choice of the Diophantine numbers for all M-ary PAM constellations since the minimum determinant of eq. (11) is equal to 1. For $n = 4$, the matrix $\mathcal{M}$ used for constructing the ST code from eq. (17) is different from the rotation matrix given in [17]. While using $\mathcal{M}$ with $\gamma = 2$ results in a ST code with non-vanishing determinant, full diversity is lost when using the rotation matrix given in [17] with $\gamma = 2$.

The coding gain of the code given in eq. (17) can be expressed as [11]:

$$g_{min}(C) \triangleq \min(\det(CC^T))^{\frac{1}{n}} = c_n(\gamma)(d_\infty(C))^{\frac{2}{n}} = c_n(\gamma)d_{\mathbb{K}}^{-\frac{1}{n}} \quad (28)$$

where g_{min} stands for the coding gain and $c_n(\gamma) = n/\sum_{i=0}^{n-1}\gamma^{\frac{2i}{n}}$ is a normalization factor insuring the same transmitted energy as in the uncoded case. A good question arises: what if there exists a TAST code associated with a certain value of γ not verifying proposition 1 and which, for a particular PAM constellation, results in a coding gain that exceeds the one given in eq. (28). For $n = 2$ and $n = 3$ transmit antennas, we can show that this can never happen since:

$$\delta_M(\gamma) \leq \delta_2(\gamma) \leq \delta_2(2) = \delta_M(2) \quad (29)$$

where $\delta_M(\gamma)$ stands for the coding gain (given in eq. (28)) of the code constructed from γ over a M-ary PAM signal set. The first inequality follows from the fact that 2-PAM is a subset of M-PAM for $M \geq 2$ and the second inequality can be checked out through computer simulations that are feasible for $n = 2$ and $n = 3$ (given the relatively small number of codewords). The last equality holds since $\gamma = 2$ verifies proposition 1. For $n > 3$, we are not sure if the last inequality can hold, even though simulations over several million elements drawn randomly among the set of 3^{n^2} elements validate such a trend. In all cases, $\gamma = 2$ is the best possible choice since the validity of another choice can not be verified. Finally, including the repetitions used to transmit one symbol, the non-balanced codes in eq. (17) and the balanced codes in eq. (26) can be expressed as: $C_n = C \otimes \mathbf{1}_{1\times N_f}$ and $C_{n,bal} = C_{bal} \otimes \mathbf{1}_{1\times N_f/n}$.

B. Inter-Pulse Coding (IPC)

In this case the matrix $B/\sqrt{N_f}$ is chosen to be unitary. From eq. (9), combining the decision variables of the pulses used to convey one symbol is equivalent to calculating:

$$C_1 = A(I_J \otimes B)(I_J \otimes B)^T = N_f A I_{JP} = N_f A \quad (30)$$

From eq. (11), eq. (13) and eq. (14), the energy-balanced version of A takes the form:

$$A(\gamma) = \sqrt{c_n(\gamma)}\text{diag}\left(\alpha_0, \ldots, \alpha_{n-1}\right) F(\gamma) \quad (31)$$

where $\alpha_i = \sigma^i(\alpha)$ when n is odd and $\alpha_i = (\sigma^i(\alpha))^{\frac{1}{2}}$ when n is even. $F(\gamma)$ is a $n \times n^2$ matrix given by:

$$F(\gamma) = \left[\text{diag}\left(f_0^0, f_{n-1}^1, \ldots, f_1^{n-1}\right), \text{diag}\left(f_1^0, f_0^1, \ldots, f_2^{n-1}\right) \cdots \text{diag}\left(f_{n-1}^0, f_{n-2}^1, \ldots, f_0^{n-1}\right)\right] \quad (32)$$

where $f_i^j = \gamma^{\frac{i}{n}}\sigma^j(k_i)$ for $i, j = 0, \ldots, n-1$.

The matrix $F(\gamma)F(\gamma)^T$ admits n eigenvalues. These eigenvalues are given by $\lambda_i = \sigma^i(\lambda)$ for $i = 0, \ldots, n-1$ and $\lambda = \sum_{i=0}^{n-1}\gamma^{\frac{2i}{n}}k_i^2$. Therefore, when α verifies eq. (15), the coding gain of $A(\gamma)$ verifies:

$$g_{min}(\gamma) = d_{\mathbb{K}}^{-\frac{1}{n}} c_n(\gamma) \min\left(\text{N}_{\mathbb{K}/\mathbb{Q}}\left(\sum_{i=0}^{n-1}\gamma^{\frac{2i}{n}}k_i^2\right)\right)^{\frac{1}{n}} \geq d_{\mathbb{K}}^{-\frac{1}{n}} c_n(\gamma) \min\left(\sum_{i=0}^{n-1}\gamma^{2i}\text{N}_{\mathbb{K}/\mathbb{Q}}(k_i)^2\right)^{\frac{1}{n}} \quad (33)$$

Consider the case $\gamma \geq 1$. The minimum of the right hand side of eq. (33) is obtained when $k_i = 0$ for $i = 1, \ldots, n-1$ and $\text{N}_{\mathbb{K}/\mathbb{Q}}(k_0) = \pm 1$. The value of γ that maximizes the coding gain is $\gamma = 1$ since $c_n(\gamma)$ is a decreasing function of γ for $\gamma \geq 1$. For $\gamma < 1$, the minimum of the right hand side of eq. (33) is obtained when $k_0 = \cdots = k_{n-2} = 0$ and $\text{N}_{\mathbb{K}/\mathbb{Q}}(k_{n-1}) = \pm 1$. Maximizing over γ results in $\gamma = 1$ since now $c_n(\gamma)\gamma^{\frac{2(n-1)}{n}}$ is an increasing function of γ. The optimal choice $\gamma = 1$ shows that since the transmitted streams are separated at the receiver side (by the use of orthogonal spreading sequences), the best strategy consists of evenly distributing the available energy among the different data streams. In this case, the coding gain achieved by IPC is equal to $d_{\mathbb{K}}^{-\frac{1}{n}}$.

Arranging the columns of eq. (31), we obtain:

$$A = \text{diag}(\alpha_0, \ldots, \alpha_{n-1})\left[\text{diag}\left(k_0, \ldots, \sigma^{n-1}(k_0)\right) \cdots \text{diag}\left(k_{n-1}, \ldots, \sigma^{n-1}(k_{n-1})\right)\right] \quad (34)$$

Equation (34) shows that encoding adjacent symbols is not needed since $k_0, \ldots, k_{n-1}$ are decoupled. Therefore, the temporal length of the codewords (J) can be chosen to be equal to one. In this case, the amplitudes of the pulses transmitted from the n antennas during each symbol duration are given by the $n \times N_f$ matrix:

$$C_n = \text{diag}\left(\mathcal{M}[a_1, \ldots, a_n]^T\right) B \quad (35)$$

where $B/\sqrt{N_f}$ is any unitary $n \times N_f$ matrix and $\mathcal{M}$ is calculated in the same way as in Subsection III-A. Since the transmitted date streams are decoupled, $\mathcal{M}$ can be any one of the rotation matrices constructed in [17] or [18].

IPC presents many advantages over ISC. From eq. (28), ISC achieves a coding gain that is $c_n(2)$ times smaller than that of IPC ($c_n(2) < 1$ for all values of n). Moreover, IPC has a lower decoding complexity since each codeword contains n rather than n^2 symbols. Finally, IPC systems have lower peak-to-average-power-ratios (PAPR) and lower decoding delays (1 symbol versus n symbol durations). However, unlike ISC that can be applied with any TH-UWB system, IPC can be applied only when $N_f \geq n$.

For multi-dimensional M-PPM-M'-PAM constellations, the codeword C with dimensions $P \times J$ (with $J = P$ for ISC and $J = 1$ for IPC) will now have the dimensions $PM \times J$ and will be noted by C_M. C_M can be calculated from eq. (17) and eq. (35) by replacing the rotation matrix $\mathcal{M}$ with $\mathcal{M} \otimes I_M$ and by replacing the scalars a_i by a_i^T where now a_i is the M-dimensional vector representation of the i-th symbol. Denote by C'_M the $P \times J$ matrix whose p-th row is equal to the sum of rows $(p-1)M+1, \ldots, pM$ of C_M. Since the rank and the determinant of $C_M^T C_M$ are greater or equal to those of $C'^T_M C'_M$

and since C'_M is equal to C, we conclude that the same value of γ will maximize the coding gain over all M-dimensional extensions of the initial M'-PAM constellation independently from the value of M.

IV. MULTIPLE ACCESS INTERFERENCE

For ISC, we fix $B^{(k)} = 1_{P\times N_f}$ for all values of k and MAI is controlled uniquely by the TH codes. For IPC, the family of unitary matrices $\{B^{(k)}\}_{k=0}^{K}$ can be properly designed in order to obtain an additional reduction in the level of MAI.

A. Designing the amplitude spreading matrices

The adopted approach consists of designing the amplitude spreading sequences and the time-hopping sequences independently from each other. Without discussing neither the benefit nor even the feasibility of jointly designed sequences, the proposed approach results in a simple design criterion that can be readily used for designing interference reducing sequences. Therefore, the TH sequences are designed as if there was no amplitude spreading and vice versa.

For synchronous systems, when no TH is used, eq. (9) can be generalized to the multi-user scheme as follows:

$$X = RA(I_J \otimes B) + \sum_{k=1}^{K} \sqrt{E_k} R^{(k)} A^{(k)} (I_J \otimes B^{(k)}) + N \quad (36)$$

where $R^{(k)}$ stands for the channel matrix of the k-th user and it is constructed in the same way as R.

From eq. (36), combining the N_f decision variables corresponding to the same symbol duration is equivalent to calculating the new decision variable X' given by:

$$\begin{aligned} X' &= X(I_J \otimes B^T) \\ &= RA + \sum_{k=1}^{K} \sqrt{E_k} R^{(k)} A^{(k)} (I_J \otimes B^{(k)} B^T) + N(I_J \otimes B^T) \end{aligned} \quad (37)$$

Based on eq. (37) and inspired from [19], we propose to construct the spreading matrices based on minimizing:

$$\max_{0\le k\le K+1} \sum_{l=0\ ;\ l\neq k}^{K+1} \det\left(B^{(k)^T} B^{(l)}\right) \quad (38)$$

This design criterion simply states that the spreading matrices of the different users must be constructed to be as orthogonal to each other as possible. Unlike the CDMA-like systems ($N_f >> 1$) where all the matrices $B^{(k)}$ are chosen to be orthogonal to each other, the limited value of N_f and the potentially large number of users sharing the channel render the totally orthogonal choice impossible. Equation (38) is equivalent to choosing the set $\{B^{(0)}, \dots, B^{(K)}\}$ that maximizes:

$$d = \min_{0\le k\le K+1} \sum_{l=0\ ;\ l\neq k}^{K+1} \prod_{p=1}^{P} \sin^2 \theta_p^{(k,l)} \quad (39)$$

where $(\theta_1^{k,l}, \dots, \theta_P^{k,l})$ are the principal angles between the two subspaces generated from the rows of $B^{(k)}$ and $B^{(l)}$.

From eq. (39), the construction of the spreading matrices depends on the number of interfering users. For example, a family of $K+1$ matrices is designed for a network with $K+1$ users. Now, if a user leaves the network, the matrices of the remaining K users must be changed since the optimal set of matrices for K users is not a subset of the set of optimal matrices for $K+1$ users. In other words, all users must update their spreading matrices whenever a user joins or leaves the network. Taking this in consideration and observing that $d \geq K d_{min}$, we propose to maximize:

$$d_{min} = \min_{0\le k\neq l\le K+1} \prod_{p=1}^{P} \sin^2 \theta_p^{(k,l)} \quad (40)$$

In this way, the inclusion property is maintained and the family of matrices can be designed based on a single parameter that corresponds to the maximum number of users whose interference must be rejected. Maximizing eq. (40) is equivalent to reducing the interference between two particular users without taking the other users in consideration. This simplification is similar to the classical approach of calculating the pairwise error probability in order to find a union bound on the performance. Another interpretation of eq. (40) is that it is the design under a worst-case scenario.

Equation (40) is nothing but the design criterion for the construction of non-coherent ST codes [20]. In what follows we fix $N_f = 2P$ which, given the number of transmit antennas, corresponds to low-dimensional spreading. Now, the approach used in [21] for the construction of such codes can be readily applied to the design of a family of matrices satisfying eq. (40). The spreading matrices are calculated according to:

$$B = I_{P\times 2P} \ \exp\begin{pmatrix} \mathbf{0}_{P\times P} & C^T \\ -C & \mathbf{0}_{P\times P} \end{pmatrix} \quad (41)$$

where C is any $P \times P$ coherent ST code satisfying the design criteria of [11]. The codewords are scaled to insure that all the principal angles in eq. (40) are smaller than $\pi/2$. In particular, C can be the code designed in eq. (17). In this way, if C is constructed from the vectors $s = [s_1, \dots, s_{P^2}]$ whose elements verify $s_i \in \{\pm 1, \dots, \pm n_u\}$, then $(2n_u)^{P^2}$ users can share the same channel. To each one of these users is given a spreading matrix corresponding to a particular value of the vector s. Moreover, since the coding gain of eq. (17) is independent from the constellation size, n_u can have large values without limiting the capability of reducing the interference level.

When amplitude spreading is combined with TH, the resulting system is expected to suffer from less interference than systems using TH exclusively. This can be compared with [9] where it was observed that randomizing the polarity of the transmitted pulses results in a better immunity against MAI even in asynchronous environments. Instead of using random sequences, we proposed here a reliable method for the construction of these sequences.

B. Performance Analysis

In order to include the effect of MAI, eq. (9) can be expressed in a more convenient way as:

$$\mathcal{X}_{(MN_fQL\times J)} = \mathcal{R}_{(MN_fQL\times PMN_f)}\big(\mathcal{B}_{(PN_f\times P)}\otimes I_M\big)C_{(PM\times J)} + \sum_{k=1}^{K}\mathcal{R}_{MAI}^{(k)}\mathcal{C}_{MAI}^{(k)} + \mathcal{N} \quad (42)$$

where C is the ST codeword whose $((p-1)M+m, j)$-th entry corresponds to the amplitude of the pulse transmitted from the p-th antenna at the m-th position of the j-th symbol duration. $\mathcal{R}$ is obtained by stacking the constituent matrices $\mathcal{R}_{q,l}$ vertically for $l = 0,\dots,L-1$ and $q = 1,\dots,Q$ where $\mathcal{R}_{q,l} = \left[I_{N_f}\otimes R_{q,l,1},\dots,I_{N_f}\otimes R_{q,l,P}\right]$. Finally, $\mathcal{B} = \text{diag}(B_1^T,\dots,B_P^T)$.

For synchronous users, $\mathcal{C}_{MAI}^{(k)} = C^{(k)}$ corresponds to the codeword transmitted by the k-th user. In this case, the matrix $\mathcal{R}_{MAI}^{(k)}$ is given by:

$$\mathcal{R}_{MAI}^{(k)} = \sqrt{E_k}\mathcal{R}^{(k)}\left(\mathcal{B}^{(k)}\otimes I_M\right) \quad (43)$$

where $\mathcal{R}^{(k)}$ is obtained by stacking the matrices $\mathcal{R}_{q,l}^{(k)}$ vertically with $\mathcal{R}_{q,l}^{(k)} = [\mathcal{R}_{q,l,1}^{(k)} \ \cdots \ \mathcal{R}_{q,l,P}^{(k)}]$ and $\mathcal{R}_{q,l,p}^{(k)} = \text{diag}\left(\mathcal{R}_{q,l,p,0}^{(k)},\dots,\mathcal{R}_{q,l,p,N_f-1}^{(k)}\right)$. $\mathcal{R}_{q,l,p,n}^{(k)}$ is a $M\times M$ matrix whose (m,m')-th element is given by:

$$\mathcal{R}_{q,l,p,n}^{(k)}(m,m') = r_{q,p}^{(k)}\left(\Delta_l + (m-m')\delta + (c(n)-c_k(n))\,T_c\right) \quad (44)$$

For asynchronous users, each time frame can interfere with two consecutive time frames. Moreover, these frames do not necessarily correspond to the same symbol duration. In this case, the expression of $\mathcal{C}_{MAI}^{(k)}$ in eq. (42) during the j-th symbol duration is given by:

$$\mathcal{C}_{MAI}^{(k)} = \left[(C^{(k)}(j-1))^T \ , \ (C^{(k)}(j))^T \ , \ (C^{(k)}(j+1))^T\right]^T \quad (45)$$

where $C^{(k)}(j) \triangleq C^{(k)}$ and $C^{(k)}(i)$ is obtained by delaying (resp. advancing) the data stream by one symbol duration for $i=j-1$ (resp. $i=j+1$).

For asynchronous users, $R_{MAI}^{(k)}$ can be expressed as:

$$\mathcal{R}_{MAI}^{(k)} = \sqrt{E_k}\mathcal{R}_{Asyn}^{(k)}\left(I_3\otimes\left(\mathcal{B}^{(k)}\otimes I_M\right)\right) \quad (46)$$

where $\mathcal{R}_{Asyn}^{(k)}$ is obtained by stacking the matrices $\mathcal{R}_{q,l}^{(k)}$ vertically with $\mathcal{R}_{q,l}^{(k)} = [\mathcal{R}_{q,l}^{(k)}(-1) \ , \ \mathcal{R}_{q,l}^{(k)}(0) \ , \ \mathcal{R}_{q,l}^{(k)}(1)]$ and $\mathcal{R}_{q,l}^{(k)}(i) = \left[\mathcal{R}_{q,l,1}^{(k)}(i) \ \cdots \ \mathcal{R}_{q,l,P}^{(k)}(i)\right]$. $\mathcal{R}_{q,l,p}^{(k)}(i)$ is a $MN_f\times MN_f$ matrix whose $(nM+m, n'M+m')$-th element is given by (for $n,n'=0,\dots,N_f-1$ and $m,m'=1,\dots,M$):

$$r_{q,p}^{(k)}\left(\Delta_l + (m-m')\delta + (c(n)-c_k(n'))\,T_c + (n-n'-iN_f)T_f - \tau^{(k)}\right) \quad (47)$$

Note that the k-th user delay can be expressed as $\tau^{(k)} = \tau_1^{(k)}T_f + \tau_2^{(k)}$ where $\tau_1^{(k)}$ is uniformly distributed over $\left[-\frac{N_f}{2} \ \frac{N_f}{2}\right]$ and $\tau_2^{(k)}$ is uniformly distributed over $\left[-\frac{T_f}{2} \ \frac{T_f}{2}\right]$. For $\tau^{(k)}\geq 0$ (resp. $\tau^{(k)}\leq 0$), $\mathcal{R}_{q,l}^{(k)}(i)$ is equal to the all zero matrix for $i=1$ (resp. $i=-1$).

In peer-to-peer scenarios, the receiver tracks the CSI of the desired user without having any specific knowledge of the channel realizations, hopping codes and spreading sequences of the interfering users. In such situations, using linear interference suppression receivers can be a good tradeoff between performance and complexity. In order to simplify the receiver structure, we choose to combine the pulses associated with each symbol prior to performing MMSE filtering.

For IPC, despreading is performed by multiplying eq. (42) by the matrix $\mathcal{D} = I_{QL}\otimes B$. After performing this multiplication, we can easily verify that the noise term remains white. Equation (42) becomes:

$$\mathcal{Y} = D\mathcal{X} = \mathcal{H}S + \sum_{k=1}^{K}\mathcal{H}^{(k)}\mathcal{S}_{MAI}^{(k)} + \mathcal{N} \quad (48)$$

where $\mathcal{H} = \mathcal{D}\mathcal{R}\left(\mathcal{B}\mathcal{M}\otimes I_M\right)$ and $\mathcal{H}^{(k)} = \mathcal{D}\mathcal{R}_{MAI}^{(k)}(\mathcal{M}\otimes I_M)$. $\mathcal{Y}$ is the new decision vector whose length is equal to $MPQLJ$ with $J=1$. $\mathcal{M}$ is any fully-diverse rotation matrix (refer to subsection III-B). We designate by $S^{(k)}$ the MP-dimensional vector representation of the information symbols transmitted by the k-th user (with $S\triangleq S^{(0)}$).

For ISC systems, $\mathcal{D} = I_{QL}\otimes B_1$ where $B_1 = 1_{1\times N_f}$. $S^{(k)}$ is now the MP^2 vector representation of the P^2 coded symbols of the k-th user (with $S\triangleq S^{(0)}$). The channel matrix $\mathcal{H}$ in eq. (48) takes the following form:

$$\mathcal{H} = \text{diag}(\underbrace{\mathcal{D}\mathcal{R}\left(\mathcal{B}\otimes I_M\right),\dots,\mathcal{D}\mathcal{R}\left(\mathcal{B}\otimes I_M\right)}_{J})(\Phi\otimes I_M) \quad (49)$$

where Φ is the matrix that describes the linear dependence between the transmitted (coded) symbols and the information symbols. In other words, Φ is the $JP\times P^2$ matrix verifying $\text{vec}(C) = \Phi S$ where $J=P$ for unbalanced systems and $J=P^2$ for balanced systems. The matrix $\mathcal{H}^{(k)}$ in eq. (48) is obtained by replacing $\mathcal{R}$ with $\mathcal{R}_{MAI}^{(k)}$ in eq. (49).

For synchronous users, $S_{MAI}^{(k)}$ in eq. (48) is given by $S_{MAI}^{(k)} = S^{(k)}$ and it has a length of MN_t where $N_t = P$ for IPC and $N_t = P^2$ for ISC. For asynchronous users, $S_{MAI}^{(k)}$ is a $3MN_t$ vector obtained by vertically concatenating a delayed version of $S^{(k)}$, $S^{(k)}$ and an advanced version of $S^{(k)}$.

The filter based on the MMSE criterion is given by:

$$\mathcal{F} = \mathcal{H}^T\left(\mathcal{H}\mathcal{H}^T + \sum_{k=1}^{K}\mathcal{H}^{(k)}\mathcal{H}^{(k)T} + \frac{N_0}{2}I\right)^{-1} \quad (50)$$

Since the receiver has no access to the CSI of the interfering users, $\mathcal{F}$ can be determined from $\mathcal{F} = E\left(S\mathcal{Y}^T\right)E\left(\mathcal{Y}\mathcal{Y}^T\right)^{-1}$ where the average is calculated over a training sequence.

Let $V = \mathcal{F}\mathcal{H}$, $V^{(k)} = \mathcal{F}\mathcal{H}^{(k)}$ and $U = \mathcal{F}\mathcal{F}^T$. For 2-PAM constellations, and conditioned on the channel realization (denoted by (R)), the probability of detecting the symbols of the p-th data stream erroneously is given by:

$$P_{e/(R)}^{(p)} = \frac{1}{2^{\mu KN_t(N_t-1)}}\sum_{S_p\in\{\pm1\}^{N_t-1}}\ \sum_{[S_{MAI}^{(1)},\dots,S_{MAI}^{(K)}]\in\{\pm1\}^{\mu KN_t}} Q\left(\frac{V_{p,p}+\overline{V}_pS_p+\sum_{k=1}^{K}V_p^{(k)}S_{MAI}^{(k)}}{\sqrt{N_0U_{p,p}/2}}\right) \quad (51)$$

where $Q(.)$ is the Gaussian tail function. X_i and $X_{i,j}$ correspond to the i-th row and the (i,j)-th element of the matrix X respectively. $\overline{V}_p$ is obtained by removing the p-th element from the p-th row of matrix V. In eq. (51), $\mu = 1$ for synchronous users and $\mu = 3$ for asynchronous users.

Assuming that after applying the MMSE filter the co-channel interference and MAI are zero-mean Gaussian random variables, eq. (51) can be approximated by:

$$P_{e/(R)}^{(p)} \approx Q\left(\frac{V_{p,p}}{\sqrt{N_0 U_{p,p}/2 + \overline{V}_p \overline{V}_p^T + \sum_{k=1}^{K} V_p^{(k)} V_p^{(k)T}}}\right) \tag{52}$$

When it is possible to calculate eq. (51), i.e. for small values of K and N_t, we realized that there is no significant difference between the values given by eq. (51) and eq. (52). This can be explained by the fact that the MMSE filter succeeds in suppressing the co-channel interference and MAI [22].

Motivated by the fact that eq. (52) is a good indicator of the error performance, we now adopt a similar approach that permits to determine an upper bound on the performance of M-PPM-2-PAM constellations. We keep the same notations as above and we denote by $V^{(p)}$ the $M \times M$ matrix obtained from the elements $V_{(p-1)M+m,(p-1)M+m'}$ of matrix V for $p, p' = 1, \dots, P$ and $m, m' = 1, \dots, M$. Supposing that a symbol $s = \pm 1$ is transmitted at the m-th position of the p-th data stream and considering that co-channel interference and MAI are Gaussian, then the outputs of the M correlators corresponding to this data stream are given by:

$$y_{m'}^{(p)} = V_{m',m}^{(p)} s + n_{m'} \tag{53}$$

for $m' = 1, \dots, M$. The term $n_{m'}$ includes the effect of noise, co-channel interference and MAI. It will be considered as a zero-mean Gaussian random variable. In this case, its variance takes the following value:

$$\sigma_{m'}^2 = \frac{N_0}{2} U_{m',m'}^{(p)} + \frac{1}{M}[\overline{V}_{(p-1)M+m'} \overline{V}_{(p-1)M+m'}^T + \sum_{k=1}^{K} V_{(p-1)M+m'}^{(k)} V_{(p-1)M+m'}^{(k)T}] \tag{54}$$

where $U^{(p)}$ is constructed in the same way as $V^{(p)}$ and $\overline{V}_i$ (for $i = (p-1)M + m'$) stands for removing the elements $(p-1)M+1, \dots, pM$ from the i-th row of the matrix V.

By applying the union bound, the conditional probability of detecting the pulse transmitted at the m-th position of the p-th data stream erroneously is bounded by:

$$P_{e/(R)}^{(p,m)} \leq \sum_{s=\pm 1} \text{prob}\left\{V_{m,m}^{(p)} s + n_m \geq -V_{m,m}^{(p)} s + n_m\right\} \text{prob}(s) + \sum_{m'=1\ ;\ m' \neq m}^{M} \text{prob}\left\{|y_{m'}^{(p)}| \geq |y_m^{(p)}|\right\} \tag{55}$$

After some manipulations, eq. (55) can be written as:

$$P_{e/(R)}^{(p,m)} \leq \mathcal{Q}_{p,m}^{(0)} + \sum_{m'=1\ ;\ m' \neq m}^{M} \mathcal{Q}_{p,m,m'}^{(1)} + \mathcal{Q}_{p,m,m'}^{(2)} - 2\mathcal{Q}_{p,m,m'}^{(1)} \mathcal{Q}_{p,m,m'}^{(2)} \tag{56}$$

where $\mathcal{Q}_{p,m,m'}^{(i)} = Q\left(\frac{V_{m,m}^{(p)} + (-1)^i V_{m',m}^{(p)}}{\sqrt{\sigma_m^2 + \sigma_{m'}^2}}\right)$ for $i = 1, 2$ and $\mathcal{Q}_{p,m}^{(0)} = Q\left(\frac{V_{m,m}^{(p)}}{\sqrt{\sigma_m^2}}\right)$.

The conditional error probability can be calculated from

$$P_{e/(R)} = \frac{1}{M} \sum_{p=1}^{P} \sum_{m=1}^{M} P_{e/(R)}^{(p,m)} \tag{57}$$

Since the performance is determined over the IEEE channel model [23] that does not lend itself to analytical solutions, the average error probability is evaluated by Monte Carlo simulations. More precisely, the error probability is determined by averaging $P_{e/(R)}$ over different realizations of the channel, the interfering users' channels, the amplitude spreading matrices and the TH sequences.

V. SIMULATIONS AND RESULTS

The pulse waveform $w(t)$ is chosen to be the second derivative of the Gaussian pulse with a duration of 0.5 ns. The PQ sub-channels of each user are generated independently according to the IEEE 802.15.3a channel model recommendation CM2 that corresponds to non-line-of-sight (NLOS) conditions [23]. The channel is held constant over one transmission block and is allowed to change independently from one block to another. The modulation delay is chosen to verify $\delta = T_w = 0.5$ ns. In order to eliminate ISI, we fix $T_f = N_c T_c + 100$ ns with $T_c = \delta$. In single-user situations, the performance is independent from the number of time frames N_f. In this case, we fix $N_f = P$ in order to render IPC and the balancing of ISC possible. In multi-user scenarios, all users are assumed to have the same transmission levels. The sphere decoder [24] is used for detection. For simplicity, we assume that the relative time delays between the signals received at the different antennas ($\varepsilon_{q,p}^{(k)}$ in eq. (2)) are negligible. In this way, the simulations highlight the diversity and multiplexing advantages of the proposed schemes independently from the relative orientations and positions of the transmit and receive arrays.

In Fig. 2, we show the performance of equilibrated, unbalanced and balanced ISCs C_{eq}, C and C_{bal} taken from eq. (21), eq. (22) and eq. (27) respectively. We compare the results of the above codes with those of IPC C_2 given in eq. (35) and with STC-scheme 1 from [1], [2]. The latter code achieves full diversity with a rate of 1 symbol PCU for all values of P. This code will be referred to as the orthogonal code (OC) in what follows. The comparison is performed at the same data rate of 2 and 4 bits PCU with signal sets having the same dimensionality for the case of 2 transmit and 2 receive antennas with a 6 fingers Rake. The difference between the coding gains of C and C_{eq} results in a gain of about 0.75 dB at high SNR. We can also notice the performance improvement introduced by balancing the ISC. Note that at 2 bits PCU, OC approaches the performance of C_{bal} at high SNR. At 4 bits PCU, and for the same number of correlators (fixed by the dimensionality of the constellation), OC must use 16-PAM resulting in important performance losses with respect to ISC and IPC. Similar results are obtained in Fig. 3 with 5 fingers Rake and 2-dimensional constellations at the rate of 4 bits PCU.

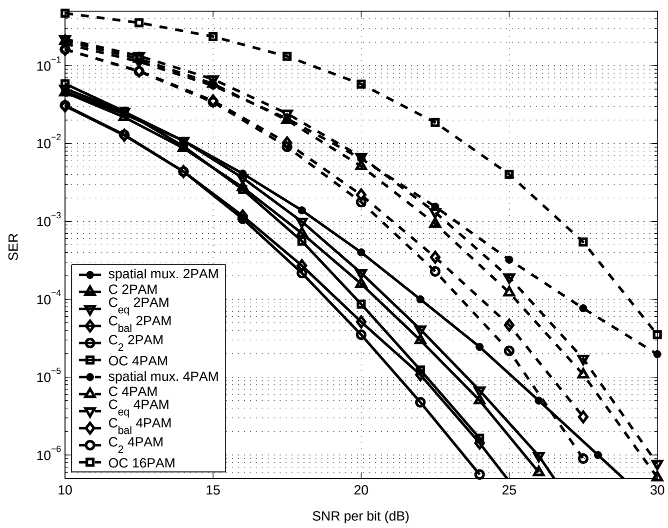

Fig. 2. Performance with 2 transmit antennas, 2 receive antennas and 6 fingers Rake. C_{eq}, C, C_{bal} and C_2 are the codes from eq. (21), eq. (22), eq. (27) and eq. (35) respectively. OC corresponds to the orthogonal code from [1].

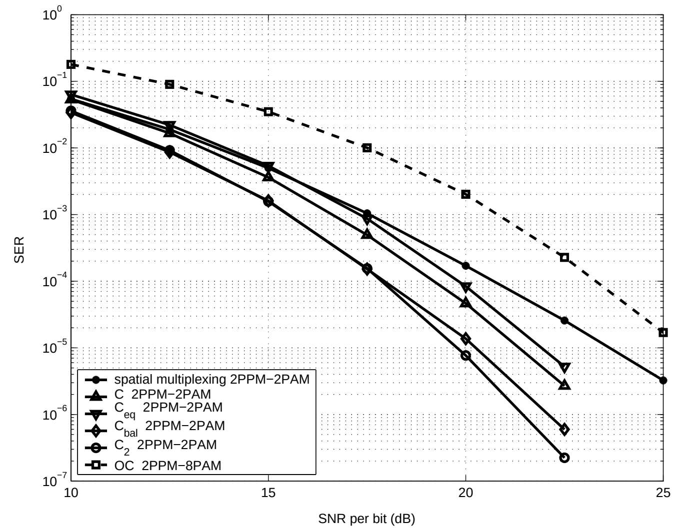

Fig. 3. Performance with P=Q=2 and 5 fingers Rake with 2-dimensional constellations at 4 bits PCU. C_{eq}, C, C_{bal} and C_2 are the codes from eq. (21), eq. (22), eq. (27) and eq. (35) respectively while OC corresponds to the orthogonal code from [1].

In Fig. 4 - Fig. 7, we fix $L = 1$, $P = Q = n$ and a data rate of n bits PCU for $n = 3, \ldots, 6$. IPC (C_n) outperforms ISC (C) in all cases. The performance of C_{bal} improves with increasing n since the large number of permutations gives it a better immunity against noise. In particular, for $n > 4$, C_{bal} outperforms C for all SNRs. OC shows bad performance even when associated with constellations that have higher dimensionality than those associated with C, C_{bal} and C_n.

Fig. 8 shows the performance with $P = Q = 2$, $N_f = 4$, $N_c = 200$, 2-PAM and $L = 5$ for $K = 10$ and $K = 15$ interfering users in a synchronous environment. The ISC C_{bal} is compared with the IPC C_2 in the two following cases. In the first case, each one of the $K+1$ users is attributed with a random unitary matrix whose elements are taken from $\{\pm 1\}$. In the second case, to each user is given a matrix form the family of 256 matrices constructed from eq. (41) using 4-PAM symbols. Results show that appropriately choosing the spreading matrices results in a gain of about 1 dB at 10^{-4} for $K = 15$. The dotted lines associated with each curve correspond to the Gaussian approximation which turns out to be accurate for all number of users.

Fig. 9 compares simulation results with the upper bound in eq. (56) for 4-PPM-2-PAM, $P = Q = 2$, $N_c = 200$, $N_f = 4$ and $L = 5$ with 6 and 11 asynchronous users. Three systems are considered. Uncoded unspreaded systems that correspond to spatial multiplexing with all of the N_f pulses having the same amplitude. Uncoded spreaded systems which is the same as before but now the matrices associated with the different users are determined from eq. (41). Coded spreaded systems which correspond to the IPC. Results show the importance of associating eq. (41) with eq. (35). On the other hand, Fig. 9 shows that the upper bound is tight for high SNRs.

Fig. 10 shows the degradation introduced by MAI when using IPC given in eq. (35) with $P = Q = 2$, $L = 7$, $N_c = 200$ and $N_f = 4$ in an asynchronous environment at a SNR of 25 dB. The random matrices whose elements are taken from $\{\pm 1\}$ and the designed matrices from eq. (41) are attributed to the different users in the same way as in the simulation setup of Fig. 8. The simulation results show that the performance gains become more significant with large number of users since in this case interference becomes more critical on the system performance.

VI. Conclusion

We discussed the construction of two families of full-rate and fully diverse ST codes that are adapted to carrier-less UWB transmissions. The first family of codes is constructed from totally real cyclic division algebras. It was shown that profiting from the repetitions used to convey one information symbol, this family of codes can be further modified to balance the energy inefficiency resulting from the use of non-unitary integers in the codewords. We also profited from the presence of these repetitions to introduce a new TH-UWB-specific version of the first family of codes referred to as inter-pulse coding schemes. Multiuser extensions were also discussed and we showed that high performance levels can be obtained when associating the ST coding schemes with the constructed amplitude spreading matrices.

Appendix

In this appendix, we use the p-adic numbers to show that -1 is not a norm in $\mathbb{K} = \mathbb{Q}(\sqrt{p})$ with $p = 3$. In other words, there is no $x \in \mathbb{K}$ verifying $\mathrm{N}_{\mathbb{K}/\mathbb{Q}}(x) = -1$. Any element $x \in \mathbb{K}$ can be written as $x = a + b\sqrt{3}$ where $a, b \in \mathbb{Q}$. We will show that the equation:

$$\mathrm{N}_{\mathbb{K}/\mathbb{Q}}(x) = a^2 - 3b^2 = -1 \tag{58}$$

has no solution in the field of 3-adic number $\mathbf{Q}_3$ and consequently has no solution in $\mathbb{K}$. Let $\mathbf{Z}_3 = \{x \in \mathbf{Q}_3, v_3(x) \geq 0\}$

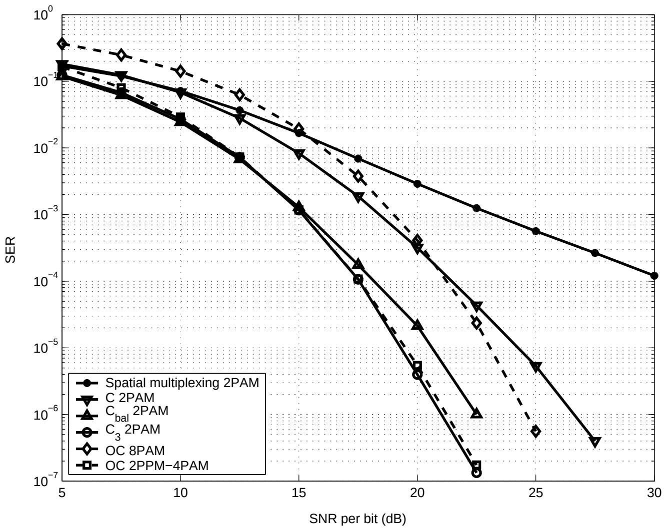

Fig. 4. Performance with 3 transmit antennas, 3 receive antennas and 1 finger Rake. C, C_{bal}, and C_3 are the codes from eq. (17), eq. (26) and eq. (35) respectively while OC corresponds to the orthogonal code from [2].

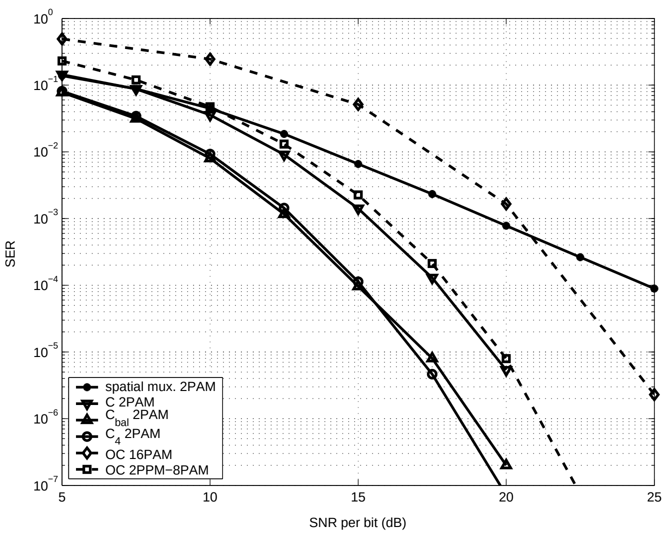

Fig. 5. Performance with 4 transmit antennas, 4 receive antennas and 1 finger Rake. C, C_{bal}, and C_4 are the codes from eq. (17), eq. (26) and eq. (35) respectively while OC corresponds to the orthogonal code from [2].

be the valuation ring of $\mathbf{Q}_3$ [25]. Using the embedding of $\mathbb{Q}$ in $\mathbf{Q}_3$, eq. (58) can be written in $\mathbf{Q}_3$ as:

$$a^2 - 3b^2 = 2 + 3x \;\; ; \;\; a, b \in \mathbb{Q}, \;\; x \in \mathbf{Z}_3 \qquad (59)$$

We take the valuations of both sides of eq. (59):

$$v_3(a^2 - 3b^2) = v_3(2 + 3x) \qquad (60)$$

to show that a and b must be in $\mathbf{Z}_3$. Since $x \in \mathbf{Z}_3$:

$$v_3(2 + 3x) \geq \min\{v_3(2), v_3(x) + 1\} = 0 \qquad (61)$$

Since both valuations are distinct ($v_3(2) = 0$ and $v_3(x) \geq 0$), we have:

$$v_3(a^2 - 3b^2) \geq \min\{2v_3(a), 2v_3(b) + 1\} = 0 \qquad (62)$$

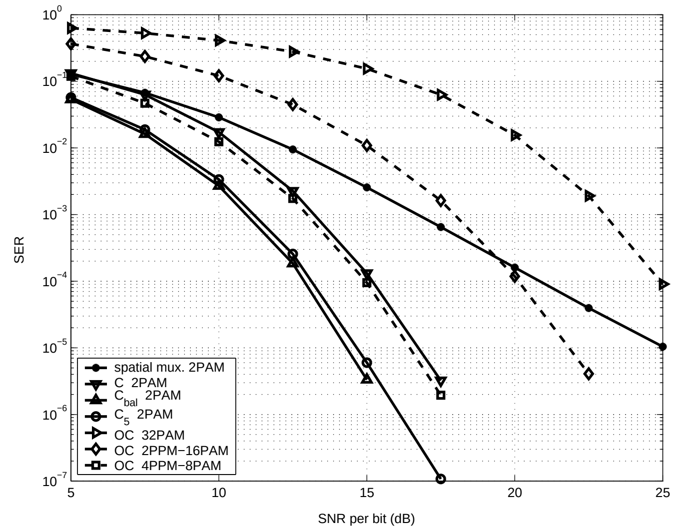

Fig. 6. Performance with 5 transmit antennas, 5 receive antennas and 1 finger Rake. C, C_{bal}, and C_5 are the codes from eq. (17), eq. (26) and eq. (35) respectively while OC corresponds to the orthogonal code from [2].

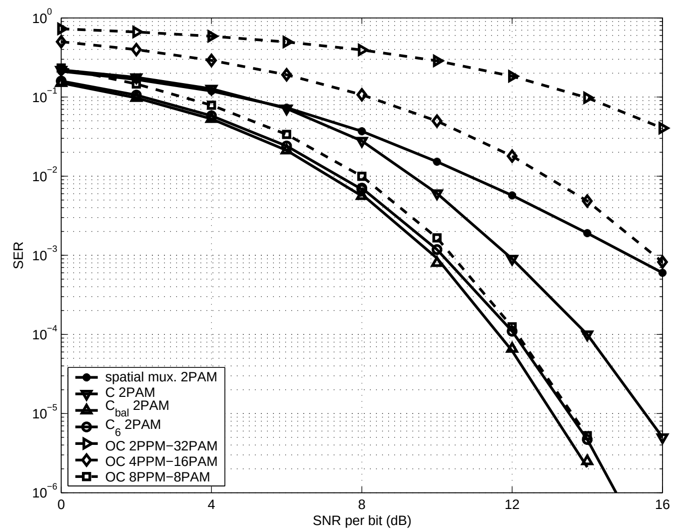

Fig. 7. Performance with 6 transmit antennas, 6 receive antennas and 1 finger Rake. C, C_{bal}, and C_6 are the codes from eq. (17), eq. (26) and eq. (35) respectively while OC corresponds to the orthogonal code from [2].

Moreover we have equality since both valuations are distinct. Equation (62) holds if $v_3(a) = 0$ which implies $a \in \mathbf{Z}_3$ and consequently $b \in \mathbf{Z}_3$. Equation (59) can be now written as:

$$a^2 - 3b^2 = 2 + 3x \;\; ; \;\; a, b, x \in \mathbf{Z}_3 \qquad (63)$$

Reducing eq. (63) modulo $3\mathbf{Z}_3$, we find that 2 should be a square in $\mathrm{GF}(3)$ which is a contradiction and therefore there are no elements $a, b \in \mathbb{Q}$ that verify eq. (58).

REFERENCES

[1] L. Yang and G. B. Giannakis, "Analog space-time coding for multi-antenna ultra-wideband transmissions," *IEEE Trans. Commun.*, vol. 52, pp. 507–517, March 2004.

[2] C. Abou-Rjeily and N. Daniele, "Performance analysis of space-time coding over ultra-wideband channels," in *Proceedings IEEE International Conference on Signal Processing and Communications*, December 2004, pp. 315–319.

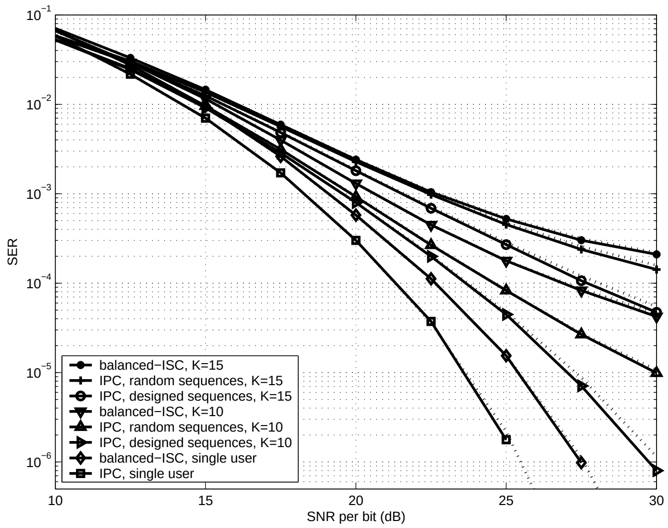

Fig. 8. Performance with $P = Q = 2$, 2-PAM and 5 fingers Rake with K interfering users in a synchronous environment. C_{bal} and C_2 are the codes from eq. (26) and eq. (35). The dotted lines associated with each curve correspond to the Gaussian approximation.

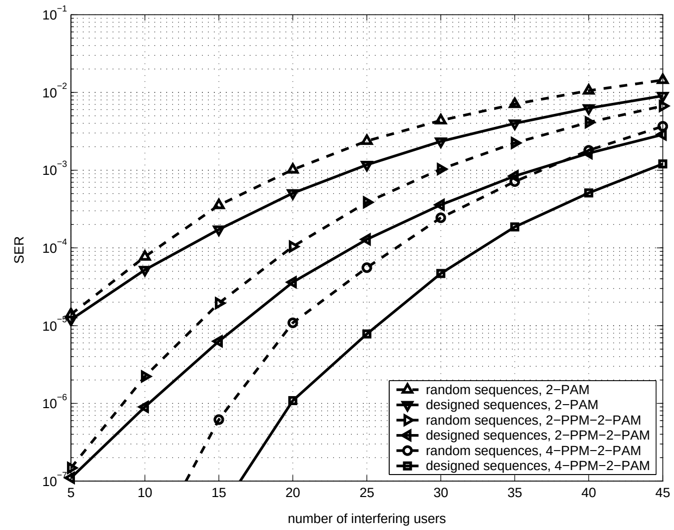

Fig. 10. Performance of IPC from eq. (35) in the presence of MAI with $P = Q = 2$ and 7 fingers Rake in an asynchronous environment at a SNR of 25 dB. The random and designed sequences correspond to random unitary matrices and to the spreading matrices constructed from eq. (41) respectively.

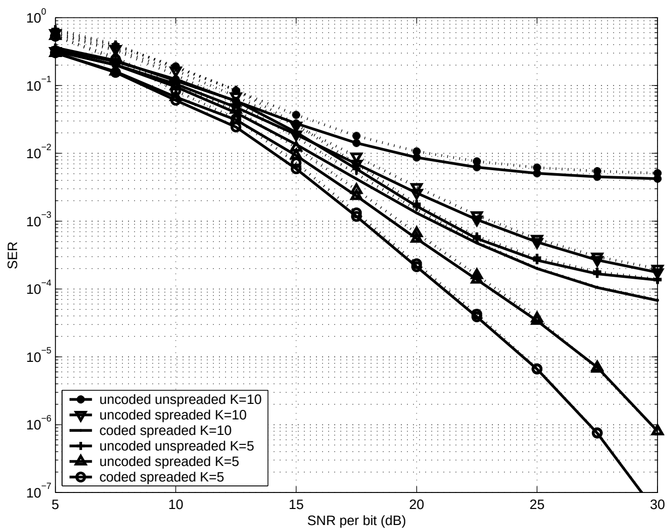

Fig. 9. Performance with $P = Q = 2$, 4-PPM-2-PAM and 5 fingers Rake with K interfering users in an asynchronous environment. The dotted lines associated with each curve correspond to the upper bound based on the Gaussian approximation.

[3] M. Weisenhorn and W. Hirt, "Performance of binary antipodal signaling over the indoor UWB MIMO channel," in *Proceedings IEEE International Conference on Communications*, vol. 4, May 2003, pp. 2872–2878.

[4] W. Siriwongpairat, M. Olfat, and K. J. R. Liu, "On the performance evaluation of TH and DS UWB MIMO systems," in *Proceedings IEEE Wireless Communications and Networking Conference*, vol. 3, March 2004, pp. 1800–1805.

[5] H. El Gamal and M. O. Damen, "Universal space-time coding," *IEEE Trans. Inform. Theory*, vol. 49, pp. 1097–1119, May 2003.

[6] B. A. Sethuraman, B. S. Rajan, and V. Shashidhar, "Full-diversity, high rate space-time block codes from division algebras," *IEEE Trans. Inform. Theory*, vol. 49, pp. 2596–2616, October 2003.

[7] ——, "STBCs using capacity achieving designs from cyclic division algebras," in *Proceedings IEEE Global Communications Conference*, vol. 4, December 2003, pp. 1957–1962.

[8] G. Rekaya, J. C. Belfiore, and E. Viterbo, "Algebraic 3×3, 4×4 and 6×6 space-time codes with non-vanishing determinants," to appear in IEEE Int. Conference on Information Theory and its Applications.

[9] S. Gezici, H. Kobayashi, V. Poor, and A. F. Molisch, "Performance evaluation of impulse radio uwb systems with pulse-based polarity randomization," *IEEE Trans. Signal Processing*, vol. 53, pp. 2537–2549, July 2005.

[10] D. Cassioli, M. Z. Win, F.Vatalaro, and A. F. Molisch, "Performance of low-complexity rake reception in a realistic UWB channel," in *Proceedings IEEE International Conference on Communications*, vol. 2, May 2002, pp. 763–767.

[11] V. Tarokh, N. Seshadri, and A. Calderbank, "Space-time codes for high data rate wireless communication : Performance criterion and code construction," *IEEE Trans. Inform. Theory*, vol. 44, pp. 744–765, 1998.

[12] J.-C. Belfiore, G. Rekaya, and E. Viterbo, "The Golden code: a 2×2 full-rate space-time code with nonvanishing determinant," *IEEE Trans. Inform. Theory*, vol. 51, no. 4, pp. 1432–1436, April 2005.

[13] F. Oggier, G. Rekaya, J.-C. Belfiore, and E. Viterbo, "Perfect space time block codes," *IEEE Trans. Inform. Theory*, submitted for publication.

[14] M. Daberkow, C. Fieker, J. Klners, M. Pohst, K. Roegner, and K. Wildanger, "Kant v4," *J. Symbolic Comp.*, vol. 24, 1997.

[15] A. Lenstra, H. Lenstra, and L. Lovász, "Factorizing polynomials with rational coefficients," *Math. Ann.*, vol. 261, pp. 515–534, 1982.

[16] J.-C. Belfiore and G. Rekaya, "Quaternionic lattices for space-time coding," in *Proceedings IEEE Information Theory Workshop*, vol. 2, April 2003, pp. 267–270.

[17] E. Bayer, F. Oggier, and E. Viterbo, "New algebraic constructions of rotated $\mathbb{Z}^n$ lattice constellations for the Rayleigh fading channel," *IEEE Trans. Inform. Theory*, vol. 50, pp. 702–714, April 2004.

[18] E. Viterbo and J. J. Boutros, "Signal space diversity: a power- and bandwidth-efficient diversity technique for the rayleigh fading channel," *IEEE Trans. Inform. Theory*, vol. 44, no. 4, pp. 1453–1467, July 1998.

[19] M. Brehler and M. K. Varanasi, "Optimum receivers and low-dimensional spreaded modulation for multiuser space-time communications," *IEEE Trans. Inform. Theory*, vol. 49, pp. 901–918, April 2003.

[20] ——, "Asymptotic error probability analysis of quadratic receivers over Rayleigh fading channels with applications to a unified analysis of coherent and noncoherent space time receivers," *IEEE Trans. Inform. Theory*, vol. 47, pp. 2383–2399, September 2001.

[21] I. Kammoun and J.-C. Belfiore, "A new family of Grassmann Space-Time codes for non-coherent MIMO systems," *IEEE Commun. Lett.*, vol. 7, pp. 528–530, November 2003.

[22] L. Zhiwei, B. Premkumar, and A. S. Madhukumar, "MMSE detection for high data rate UWB MIMO systems," in *Proceedings IEEE Vehicular Technology Conference*, vol. 2, September 2004, pp. 1463–1467.

[23] J. Foerster, "Channel modeling sub-committee Report Final," Technical report IEEE 802.15-02/490, IEEE 802.15.3a Wireless Personal Area Networks, 2002.

[24] M. O. Damen, H. El-Gamal, and G. Caire, “On maximum-likelihood detection and the search for the closest lattice point,” *IEEE Trans. Inform. Theory*, vol. 49, pp. 2389–2402, October 2003.

[25] F. Q. Gouvêa, *P-Adic Numbers: An introduction, 2nd ed.* Berlin, Germany: Springer-Verlag, 1997.